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Twistor space structure of one-loop amplitudes in gauge theory

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Abstract: We analyze the twistor space structure of certain one-loop amplitudes in gauge theory. For some amplitudes, we find decompositions that make the twistor structure manifest; for others, we explore the twistor space structure by finding differential equations that the amplitudes obey.

Keywords: Gauge Symmetry, Supersymmetry and Duality, Extended Supersymmetry, NLO Computations.

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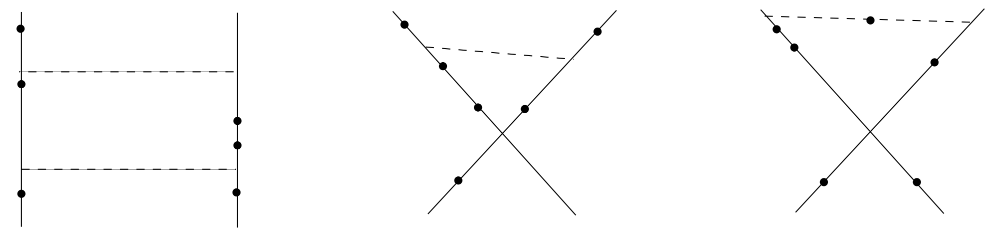
Contents

1. Introduction

Perturbative gauge theory has many remarkable properties, among them the surprising accessability and elegant structure of certain one-loop S-matrix elements [1, 2]. Some of the unexpected simplicity of perturbative gauge theory can be explained by reinterpreting this subject in terms of a topological string theory with twistor space as the target. This was proposed in [3], where more detailed references concerning perturbative gauge theory and twistor space can be found.



(a) (b) (c)

Figure 1: Shown here are twistor configurations that we find contribute to one-loop supersymmetric MHV amplitudes. In (a), all gluons are inserted on a pair of disjoint lines. In (b), all gluons are inserted on a pair of intersecting lines. (c) is just like (b) except that one gluon is inserted not on the pair of intersecting lines but somewhere else in the plane containing the two lines. In the figures, dashed lines indicate twistor space propagators whose presence we conjecture, though it is not directly revealed by calculations in this paper.

Even if one does not know a twistor-string theory appropriate for computing a given scattering amplitude, or does not understand it properly, one can, as explained in section 3 of [3], gain some insight by studying the differential equations that the scattering amplitudes obey. With this in mind, we have undertaken a detailed study of differential equations obeyed by tree level and one-loop scattering amplitudes in gauge theories with varying degrees of supersymmetry. The tree level results suggested an interpretation via "MHV tree diagrams" that we have presented separately [4]. The present paper aims to explain the one-loop results.

For gluon scattering at tree level (in renormalizable gauge theories), supersymmetry does not matter. For loop amplitudes, it does. Not all one-loop amplitudes have been computed. Roughly speaking, in this paper we consider nearly all of the available one-loop gluon scattering amplitudes with massless internal lines. We study MHV (maximally helicity violating) one-loop amplitudes in theories with $\mathcal{N} = 4$ or $\mathcal{N} = 1$ supersymmetry [1, 2], and certain nonsupersymmetric one-loop amplitudes [2].

Our results for the supersymmetric one-loop MHV amplitudes are qualitatively similar to what one would guess based on the twistor-string conjecture in [3], but there is an apparent discrepancy, whose meaning will be clarified in a separate paper. From [3], one would anticipate two possible types of twistor space contribution to a one-loop n-gluon MHV scattering process. In one configuration, sketched in figure 1a, all n gluons are supported on a pair of lines in twistor space; the lines are connected by two twistor propagators. In the second configuration, sketched in 1b, all n gluons are supported on a curve C of genus zero and degree two; there is also a twistor space propagator connecting this curve to itself. Our study of the differential equations, however, has revealed that C reduces to a pair of intersecting lines, and that the conditions in 1b need to be somewhat relaxed. The supersymmetric one-loop MHV scattering amplitudes actually appear to receive contributions of the type indicated in 1c, with only $n-1$ of the gluons contained in the two lines. The two intersecting lines are automatically contained in a $\mathbb{CP}^2 \subset \mathbb{CP}^3$, and the n^{th} gluon is contained in this $\mathbb{CP}^2$ (with "derivative of a delta function" support).

We do not know what kind of twistor-string theory would generate this structure, but we hope that our result may serve as a useful clue. There may be an important difference between pure super Yang-Mills theory (which we study here), and super Yang-Mills theory coupled to conformal supergravity, which [5] is described by currently known forms of twistor-string theory.

In section 2, we briefly review the use of differential equations to investigate the twistor space structure of scattering amplitudes, and explain how analysis of these equations served as a clue to the description of Yang-Mills tree amplitudes via MHV tree diagrams [4]. In section 3, we study the one-loop MHV amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory. Our main result is a noncovariant decomposition of the amplitude that makes almost manifest the differential equations obeyed by the amplitude and its twistor space structure. The decomposition is somewhat similar to the one we found for tree amplitudes in our previous paper, but in contrast to that case, we are unfortunately not able to give a simple explanation of what the pieces mean. In section 4, we study the one-loop MHV amplitudes in $\mathcal{N} = 1$ super Yang-Mills theory. We find that, at least for one-loop MHV amplitudes, the twistor space structure for $\mathcal{N} = 1$ seems to be nearly the same as for $\mathcal{N} = 4$. In section 5, we study some nonsupersymmetric one-loop amplitudes. Again, analysis of differential equations suggests a twistor space structure that is surprisingly similar to the supersymmetric case. The most important difference may be that in nonsupersymmetric gauge theories, there is a one-loop amplitude for gluons all of positive helicity that must be included as a new building block alongside the MHV tree amplitudes.

As we will explain in a separate paper, the configuration of figure $1c$ arises, in a certain sense, from a holomorphic anomaly in evaluating the amplitude derived from a twistor space configuration of type $1(a)$. When this is taken into account, one can possibly salvage the naive twistor space viewpoint of figures $1a$ and $1b$.

2. Review of differential equations and tree level amplitudes

Scattering amplitudes are usually described for particles of definite momentum p_μ. For a massless particle, one can factor the momentum in terms of spinors; setting $p_{a\dot{a}} = \sigma^\mu_{a\dot{a}} p_\mu$, one has $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}$, with λ_a and $\widetilde{\lambda}_{\dot{a}}$ being spinors of, respectively, positive and negative helicity. Spinor inner products are defined by $\langle \lambda, \lambda' \rangle = \epsilon_{ab} \lambda^a \lambda'^b$, $[\widetilde{\lambda}, \widetilde{\lambda}'] = \epsilon_{\dot{a}\dot{b}} \widetilde{\lambda}^{\dot{a}} \widetilde{\lambda}'^{\dot{b}}$. We also write $\langle i, j \rangle$ for $\langle \lambda_i, \lambda_j \rangle$, and $[i, j]$ for $[\widetilde{\lambda}_i, \widetilde{\lambda}_j]$. For further details of spinor notation, see section 2 of [3].

Twistor space [6] is introduced by a Fourier transform from $\widetilde{\lambda}^{\dot{a}}$ to a new variable $\mu^{\dot{a}}$. (See [3] for a detailed description of this Fourier transform.) One interprets $Z^I = (\lambda^a, \mu^{\dot{a}})$ as homogeneous coordinates of a complex projective space $\mathbb{CP}^3$ that is known as twistor space. (Twistor space also has a supersymmetric extension, but this will not be needed here.)

In a twistor description, instead of describing the i^{th} external massless particle in an n-particle scattering amplitude by its momentum $p_i^{a\dot{a}} = \lambda_i^a \widetilde{\lambda}_i^{\dot{a}}$, one associates it with a point Z_i in twistor space with homogeneous coordinates $Z_i^I = (\lambda_i^a, \mu_i^{\dot{a}})$. The twistor space scattering amplitude is a function (or $(0,n)$-form) $\widetilde{A}(Z_1^I, \ldots, Z_n^I)$ on $(\mathbb{CP}^3)^n$, one copy of $\mathbb{CP}^3$ for each external particle. By "determining the twistor space structure" of the scattering amplitude, we mean determining the smallest algebraic subspace of $(\mathbb{CP}^3)^n$ on which $\widetilde{A}(Z_1^I, \ldots, Z_n^I)$ is supported. The answer is interesting if this subspace has a small dimension and, hopefully, a simple string theory interpretation.

2.1 Building blocks of differential equations

A variety of conditions on collections of points in $\mathbb{CP}^3$ were considered in section 3 of [3]. Happily, in the present paper we need only the simplest of those conditions. Given three points $P_i, P_j, P_k \in \mathbb{CP}^3$ with homogeneous coordinates Z_i^I, Z_j^I, and Z_k^I, the condition that they lie on a "line," that is a linearly embedded copy of $\mathbb{CP}^1$, is that $F_{ijkL} = 0$, where

$$F_{ijkL} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K \,. \tag{2.1}$$

And given four points $P_i, P_j, P_k, P_l \in \mathbb{CP}^3$, the condition that they are all contained in a "plane," that is a linearly embedded copy of $\mathbb{CP}^2 \subset \mathbb{CP}^3$, is that $K_{ijkl} = 0$, where

$$K_{ijkl} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L \,. \tag{2.2}$$

Suppose that we are presented with the momentum space version of a scattering amplitude. We describe it in terms of spinors as a function $A(\lambda_1^a, \widetilde{\lambda}_1^{\dot a}; \ldots ; \lambda_n^a, \widetilde{\lambda}_n^{\dot a})$. The condition that the equivalent twistor space amplitude has support where the points i, j, k are collinear is that $F_{ijkL}A = 0$. Here F_{ijkL} is interpreted as a differential operator (acting on a function of $\lambda, \widetilde{\lambda}$ rather than λ, μ) via $\mu \to i\partial/\partial\lambda$. It is often useful to abbreviate the statement that $F_{ijkL}A = 0$ for all L and write simply $F_{ijk}A = 0$. Likewise, the condition that the twistor space amplitude has support where the points i, j, k, l are coplanar is that $K_{ijkl}A = 0$, where K_{ijkl} is similarly interpreted as a differential operator. This process of interpreting a function of twistor coordinates as a differential operator on momentum variables is implemented in many examples in section 3 of [3].

We can give a few simple criteria for an amplitude to be annihilated by the collinear operator F_{ijkL}. Setting $L = \dot a$, the operator $F_{ijk\dot a}$ is concretely

$$\langle i, j\rangle \frac{\partial}{\partial \widetilde{\lambda}_k^{\dot a}} + \langle j, k\rangle \frac{\partial}{\partial \widetilde{\lambda}_i^{\dot a}} + \langle k, i\rangle \frac{\partial}{\partial \widetilde{\lambda}_j^{\dot a}} \,. \tag{2.3}$$

Obviously, this operator annihilates any scattering amplitude A that depends on the λ's but is independent of the $\widetilde{\lambda}$'s. This is likewise true of the $L = a$ components of F_{ijkL}, which are homogeneous and quadratic in $\partial/\partial\widetilde{\lambda}$. So if the scattering amplitude is independent of $\widetilde{\lambda}_i$, $\widetilde{\lambda}_j$, and $\widetilde{\lambda}_k$, then particles i, j, k are supported on a line in twistor space.

An amplitude that depends only on the λ's and not the $\widetilde{\lambda}$'s is often said to be "holomorphic." The motivation for this terminology comes from considering physical scattering amplitudes in Minkowski spacetime. For real momenta in Lorentz signature, $\widetilde{\lambda}$ is the complex conjugate of λ (up to a sign that depends on the sign of the energy), so in real Minkowski spacetime, an amplitude is holomorphic in λ precisely if it is independent of $\widetilde{\lambda}$.

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More generally, if the scattering amplitude $A(\lambda,\widetilde{\lambda})$ is polynomial in $\widetilde{\lambda}_i$, $\widetilde{\lambda}_j$, and $\widetilde{\lambda}_k$, then it is annihilated by F^s_{ijk} (that is, by all components of $F_{ijkL_1}F_{ijkL_2}\dots F_{ijkL_s}$) for some integer s. In this situation, as explained in [3], particles i,j, and k are still supported on a line in twistor space, but now with "derivative of a delta function" (or multiple derivative of a delta function) support. A recent computation [5] of tree level MHV scattering involving supergravitons gives one example of how such polynomial dependence on $\widetilde{\lambda}$'s can arise; in that example, the factors of $\widetilde{\lambda}$ come from the structure of the vertex operators.

All this has another important generalization. Let $P^{a\dot{a}} = p_i^{a\dot{a}} + p_j^{a\dot{a}} + p_k^{a\dot{a}}$ be the sum of the momenta of particles i,j, and k. Then F_{ijk} annihilates any amplitude $A(\lambda_i,\lambda_j,\lambda_k;P)$ that depends on $\widetilde{\lambda}_i$, $\widetilde{\lambda}_j$, and $\widetilde{\lambda}_k$ only via P. For example, to verify that $F_{ijk\dot{a}}A=0$, after using the chain rule, we need

$$\langle\lambda_i,\lambda_j\rangle\lambda_k^a + \langle\lambda_j,\lambda_k\rangle\lambda_i^a + \langle\lambda_k,\lambda_i\rangle\lambda_j^a = 0\,. \tag{2.4}$$

This identity holds because the quantity on the left takes values in a two-dimensional vector space, and is trilinear and antisymmetric in the three vectors λ_i, λ_j, and λ_k.

2.2 Examples

Now let us discuss some examples. In all cases, we consider single-trace amplitudes with n gluons in cyclic order $123,\dots,n$. We begin with the five gluon tree amplitude $A(\lambda_1,\widetilde{\lambda}_1;\dots;\lambda_5,\widetilde{\lambda}_5)$ with three gluons of negative helicity. Differential equations obeyed by this amplitude were described in section 3 of [3].

There are two types of equation. One asserts that the amplitude is supported on configurations for which three consecutive points out of the five are contained in a "line," that is a $\mathbb{CP}^1$. The other two points are automatically on a line (as there is a straight line through any two points), so the five points are actually on a union of two lines. The system of differential equations which exhibits this fact is

$$\prod_{k=1}^{5} W_{kI_k}A = 0, \tag{2.5}$$

where $W_{kI} = F_{k-1,k,k+1I}$ annihilates amplitudes for which particles $k-1,k$, and $k+1$ are collinear. Eqn. (2.5) holds for arbitrary choices of the indices I_k, and we abbreviate it by writing $\prod_{k=1}^5 W_k A = 0$.

In twistor space, where the W_k are simply multiplication operators, the assertion that A is annihilated by $\prod_{k=1}^5 W_k$ means that it is supported on the subset of $(\mathbb{CP}^3)^5$ on which at least one of the W_k vanishes, that is, on which at least three consecutive gluons are collinear.

The other differential equation obeyed by A was found to be that $K_{ijkl}A=0$ for all i,j,k, and l. This asserts that the amplitude is supported on configurations of five points that lie in a common $\mathbb{CP}^2$. (The equation $K_{ijkl}A=0$ is actually only valid for generic momenta; delta function terms enter this equation, as was anticipated to some extent in section 3 of [3] and as we explain below.)

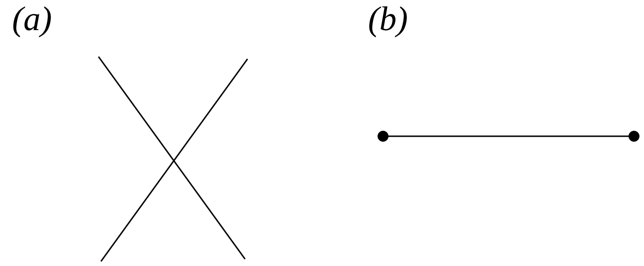

Figure 2: a A pair of intersecting lines. (b) The quiver corresponding to (a). Each vertex in the quiver represents a line; two vertices are connected if and only if the lines intersection

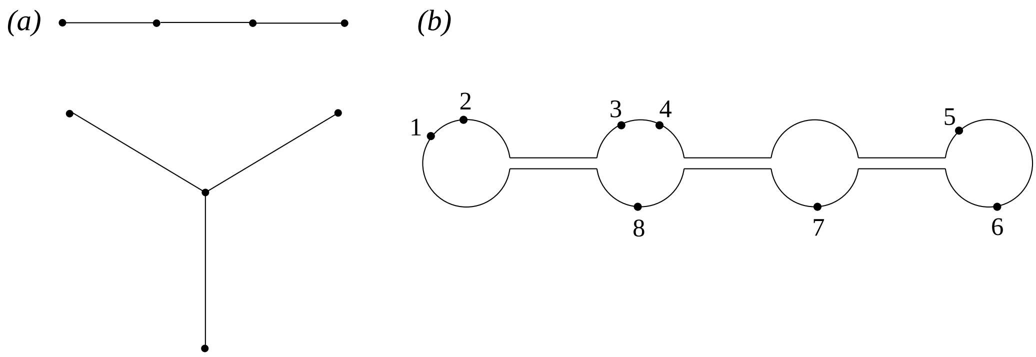

Figure 3: (a) The two tree-level quivers with four vertices. (b) An arrangement of gluons corresponding to the first quiver in (a). Each vertex in the quiver is represented by a disc, a line joining vertices is represented by a thin strip connecting two discs, and the gluons are arranged on the boundaries of the discs. Shown is an arrangement contributing to a single-trace amplitude with eight gluons.

Two lines in three-space intersect if and only if they are contained in a common plane, which in the present context means a common $\mathbb{CP}^2$. So the two sets of equations, taken together, mean that the five gluons are inserted on a pair of intersecting lines. The relevant configurations are indicated in figure 2a.

We associate the configuration of 2a with a certain "quiver." A quiver is just a graph containing points or vertices, some of which are connected by lines. We restrict ourselves to connected quivers, and for quivers related to tree diagrams, we want graphs that contain no closed loops. To make a quiver from a configuration of lines in twistor space, we draw a vertex for every line, and we connect two vertices if and only if the corresponding lines intersection Thus, the configuration in 2a corresponds to the simple quiver in 2b.

Now we can describe the results of our study of differential equations obeyed by Yang-Mills tree amplitudes with many gluons: the tree amplitudes in general are supported on quivers in twistor space. In the case of an n-gluon amplitude with q gluons of negative helicity, the quiver is constructed from $d = q - 1$ intersecting lines. In general, all possible topologies for the quivers must be included. (The first case with more than one possible quiver is $d = 4$, where there are two possibilities, as indicated in figure 3a.) For each quiver, one must sum over different arrangements of particles among the various lines (or $\mathbb{CP}^1$'s) in the quiver. The allowed arrangements can be motivated by a hypothetical duality between

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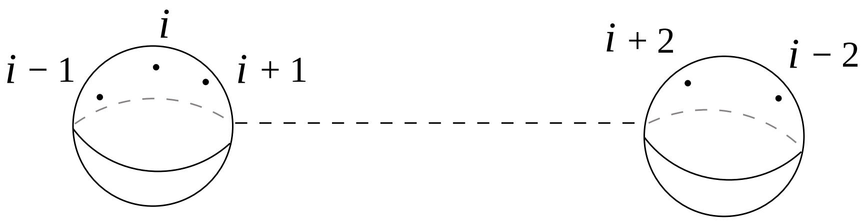

Figure 4: Two "lines," that is two $\mathbb{CP}^1$'s, with gluon insertions on them, connected by a twistor space propagator and contributing to a tree-level five-gluon amplitude.

twistor-string theory and a type-I string theory based on Chan-Paton factors. We imagine replacing each $\mathbb{CP}^1$ by a small disc, and each intersection of $\mathbb{CP}^1$'s by a thin strip connecting the discs. Finally, we sum over all arrangements of external gluons on the boundaries of the discs that are compatible with the cyclic ordering $123,\ldots,n$. For example, for the case that the number of gluons is $n = 8$, one possible arrangement corresponding to the first quiver in $3a$ is shown in $3b$.

To show that the amplitude is supported on the quiver, we do the following. Let T be the set of possible arrangements on possible quivers, with the quivers and arrangements constructed by the rules of the last paragraph. For each $t \in T$, pick an operator $\mathcal{D}_t$ that should annihilate a configuration in twistor space associated with that arrangement. For example, if t corresponds to the arrangement of eight gluons in $3b$, we could take $\mathcal{D}_t$ to be F_{348} or K_{1234} (since in this configuration, particles $3, 4$, and 8 are contained in a line, and particles 1,2,3, and 4 are contained in a pair of intersecting lines and hence in a plane). Then the claim is that

$$\prod_{t\in T} \mathcal{D}_t A = 0\,. \tag{2.6}$$

This should hold for all choices of $\mathcal{D}_t$.

Even for modest values of n and q, the differential equations in (2.6) are very numerous (as there are many choices of the $\mathcal{D}_t$) and of very high order. Nevertheless, for a certain range of n and q, sufficient to be convincing, we established with some computer assistance that Yang-Mills tree amplitudes obey (2.6) and are annihilated by no other differential operators that can be expressed as products of a comparably small number of F's and K's. The quiver picture is more restrictive than what we originally anticipated based on [3], and was discovered by trial and error with the differential equations.

Happily, we need not explain any further the details of these tree level differential equations, because there is a more transparent way to understand the quiver picture: it motivated the concept of MHV tree diagrams [4], which in fact make the quiver picture manifest.

To illustrate the point, we reconsider the five-gluon amplitude with three gluons of negative helicity. The extension of this discussion to general MHV tree diagrams and the associated quivers will hopefully be clear. In the approach via MHV tree diagrams, the five-gluon amplitude is obtained as the sum of five MHV tree diagrams, one of which is indicated in figure 4. Each diagram contains a pair of MHV vertices connected by a propagator. One vertex contains (say) gluons $i-1$, i, and $i+1$ and the other contains the other two gluons. The two MHV vertices form the vertices of a quiver (which is simply the quiver of figure 2a), and the choice of how to assign particles to the different MHV vertices corresponds precisely to the choice of arrangement of particles on this quiver.

The propagator in the MHV tree diagram is $1/P^2$, where $P = p_{i-1}+p_i+p_{i+1}$ is the total momentum flowing between the vertices. We claim that the amplitude of this particular MHV tree diagram is supported on configurations in which gluons $i-1$, i, and $i+1$ are collinear. Indeed, the amplitude is annihilated by W_i since the criterion of section 2.1 is satisfied: each MHV vertex depends on the $\widetilde{\lambda}$'s only through P, while the propagator depends only on P^2. The five-gluon amplitude is a sum of five MHV tree diagrams each of which is annihilated by one of the W_i, so the total amplitude is annihilated by the product $W_1W_2W_3W_4W_5$, a statement that is part of the quiver picture.

The remainder of the quiver picture for the five-gluon amplitude is the assertion that this amplitude is annihilated by the operators K_{ijkl} that measure coplanarity of four points in twistor space. Each MHV diagram separately has this property; it is instructive to verify this directly by writing K as a differential operator that acts on the amplitude derived from an MHV tree diagram. But another approach to explaining the statement is more illuminating, and also shows the limitations of the quiver picture.

To get the physical scattering amplitudes, the propagator in an MHV tree diagram should be $i/(P^2+i\epsilon)$. In coordinate space, the Fourier transform of this function is the Feynman propagator $D_F(x)$, which has non-zero support inside and outside the light cone (for example, see section 2.4 of [7]). However, with a different $i\epsilon$ prescription, the Fourier transform of $1/P^2$ in four dimensions has its support on the light cone.

To explain this, we start with a massless scalar field ϕ. The retarded propagator is defined as $D_R(x) = \vartheta(x^0)\langle\Omega|[\phi(x),\phi(0)]|\Omega\rangle$, where $|\Omega\rangle$ is the vacuum state, and $\vartheta(x^0)$ is equal to 1 for $x^0 \geq 0$ and vanishes for $x^0 < 0$. The retarded propagator in n dimensions obeys

$$\Box D_R(x) = -i\delta^n(x)\,, \tag{2.7}$$

where $\Box$ is the massless wave operator. In any dimension and for a particle of any mass, $D_R(x)$ vanishes outside the light cone, by virtue of causality, and for $x^0 < 0$, because of the factor of $\vartheta(x^0)$ in its definition. For a massless particle in an even spacetime dimension, $D_R(x)$ also vanishes *inside* the light cone; it is supported entirely on the future light cone. (In four dimensions, this statement is an aspect of Huygen's principle; the light signal observed at a given point in spacetime depends only on the sources *on* the past light cone, not on sources *inside* the past light cone.)

By virtue of (2.7), the Fourier transform of $D_R(x)$ is i/P^2, with some way of treating the singularity at $P^2 = 0$. Therefore, if we use the retarded propagator[1] in an MHV tree diagram, rather than the Feynman propagator, the propagation will occur only on the light cone. By contrast, the Feynman propagator $D_F(x)$, whose Fourier transform is $i/(P^2+i\epsilon)$, describes propagation inside, outside, or on the light cone.

[1] This and subsequent statements also hold if we use the advanced propagator or a half-retarded, half-advanced propagator.

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The vertices in the MHV tree diagram of figure 4 represent lines (or $\mathbb{CP}^1$'s) in twistor space that correspond to points in Minkowski space. From what we have just said, if we use the retarded propagator, these points are at lightlike separation in Minkowski space. In the twistor transform, points in Minkowski space that are at lightlike separation correspond to $\mathbb{CP}^1$'s that intersection.[2] So if we use the retarded propagator, the $\mathbb{CP}^1$'s corresponding to the vertices intersect and are thus coplanar.

Since the correct physical amplitude is derived from the Feynman propagator rather than the retarded propagator, one should hesitate to claim that the lines intersect and the amplitudes are supported on a quiver. The two propagators differ in momentum space by delta function terms supported at $P^2 = 0$ — terms which we usually overlook in studying the differential equations. Our argument shows that if (and only if) one ignores such delta function terms in evaluating $K_{ijkl}A$, one will get $K_{ijkl}A = 0$.

In [8], Feynman decomposes one-loop amplitudes by expressing the Feynman propagator as the retarded propagator plus a function that is supported on-shell. We believe that this decomposition may be related to the twistor space decomposition displayed in figure 1.

Conclusion. We have explained why and to what extent the quiver picture is true. Despite its limitations, the quiver picture was an important clue to the understanding of MHV tree diagrams that we have presented elsewhere [4]. In the rest of this paper, we attempt to determine the one-loop analog of the quiver picture (at least for some classes of one-loop amplitudes), hoping that this will similarly serve as a useful clue to a better understanding of twistor-string theory.

3. $\mathcal{N} = 4$ one-loop MHV amplitudes

3.1 Description of the amplitudes

One-loop amplitudes with external gluons are notoriously difficult to compute by evaluating Feynman diagrams directly. Often a simpler problem is to find the discontinuities along the branch cuts of the amplitude, in other words, the unitarity cuts. In general, knowing the cuts does not fix the amplitude, for there can be single-valued functions which lack cuts. However, Bern, Dixon, Dunbar, and Kosower [1] have shown that some amplitudes in gauge theories can be completely determined by their four-dimensional cuts; these are called "cut-constructible" amplitudes.

Remarkably, all $\mathcal{N} = 4$ one-loop amplitudes turn out to be "cut-constructible" [1]. This result made it possible to compute, at one-loop order, MHV amplitudes with any number of gluons [1], and also the six-gluon amplitudes with any helicities [2]. In this section, we analyze the twistor space structure of the $\mathcal{N} = 4$ one-loop MHV amplitudes. More precisely, we consider the leading-color partial amplitudes, i.e., the single-trace contributions. However, it turns out that multi-trace partial amplitudes are given as combinations of the leading-color partial amplitudes (see section 7 of [1]). Therefore our conclusions about the leading-color partial amplitudes are also valid for the total amplitude.

[2] Let one $\mathbb{CP}^1$ be given by $\mu^{\dot{a}} = x^{a\dot{a}}\lambda_a$ and the other by $\mu^{\dot{a}} = y^{a\dot{a}}\lambda_a$. The difference $y - x$ is lightlike if and only if $\det(y_{a\dot{a}} - x_{a\dot{a}}) = 0$, which is the condition for existence of λ such that $(y^{a\dot{a}} - x^{a\dot{a}})\lambda_a = 0$. Precisely when that is so, the two $\mathbb{CP}^1$'s intersect at that value of λ and $\mu^{\dot{a}} = x^{a\dot{a}}\lambda_a = y^{a\dot{a}}\lambda_a$.

MHV one-loop amplitudes (unlike more general one-loop amplitudes in the $\mathcal{N}=4$ theory) have a simple dependence on the helicity of the external gluons. In fact, one-loop MHV n-gluon amplitudes $A_n^{1-\mathrm{loop}}$ can be written

$$A_n^{1-\mathrm{loop}} = A_n^{\mathrm{tree}} V_n \,. \tag{3.1}$$

A_n^{tree} is the familiar Parke-Taylor tree level MHV amplitude [9], which contains the information about the helicities, while V_n is a universal one-loop function independent of which two gluons have negative helicity.[3]

Because A_n^{tree} is a holomorphic function of positive chirality spinor variables λ_i, $i=1,\ldots,n$, it will really not affect our analysis. The key point is to study the one-loop function V_n.

The universality of V_n implies that it is invariant under cyclic permutations of the gluons. The evaluation of V_n [1] shows that it can be naturally written as a sum of terms in which r consecutive gluons, say $i, i+1, \ldots, i+r-1$ (for some i and r), combine together in a certain sense, as do $n-2-r$ other gluons, which in this case are $i+r+1, \ldots, i-2$. Two gluons, labeled $i-1$ and $i+r$, remain uncombined. (For a pictorial representation, see appendix A.) The resulting contribution is symmetric under exchange of the two sets of "combined" gluons, so we can restrict to $r \leq (n-2)/2$. V_n is written as a sum over such choices of i and r:

$$V_n = \sum_{i=1}^{n} \sum_{r=1}^{\left[\frac{n}{2}\right]-1} \left(1 - \frac{1}{2}\delta_{n/2-1,r}\right) F^{\mathrm{2m\ e}}_{n:r;i} \,. \tag{3.2}$$

Here $F^{\mathrm{2m\ e}}_{n:r;i}$ is known as the box function and it is given by

$$\begin{aligned} F^{\mathrm{2m\ e}}_{n:r;i} = &-\frac{1}{\epsilon^2}\left[(-t_{i-1}^{[r+1]})^{-\epsilon} + (-t_i^{[r+1]})^{-\epsilon} - (-t_i^{[r]})^{-\epsilon} - (-t_{i+r+1}^{[n-r-2]})^{-\epsilon}\right] + \\ &+\mathrm{Li}_2\left(1 - \frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\right) + \mathrm{Li}_2\left(1 - \frac{t_i^{[r]}}{t_i^{[r+1]}}\right) + \\ &+\mathrm{Li}_2\left(1 - \frac{t_{i+r+1}^{[n-r-2]}}{t_{i-1}^{[r+1]}}\right) + \mathrm{Li}_2\left(1 - \frac{t_{i+r+1}^{[n-r-2]}}{t_i^{[r+1]}}\right) - \\ &-\mathrm{Li}_2\left(1 - \frac{t_i^{[r]} t_{i+r+1}^{[n-r-2]}}{t_{i-1}^{[r+1]} t_i^{[r+1]}}\right) + \frac{1}{2}\log^2\left(\frac{t_{i-1}^{[r+1]}}{t_i^{[r+1]}}\right) . \end{aligned} \tag{3.3}$$

Here we define $t_i^{[k]} = (p_i + p_{i+1} + \cdots + p_{i+k-1})^2$. An explanation of why this function is called the box function, as well as the reason for the superscript (2m e), can be found in appendix A.

In this formula, $\mathrm{Li}_2(x)$ denotes the dilogarithm function as defined by Euler, i.e., $\mathrm{Li}_2(x) = -\int_0^x \log(1-z)\, dz/z$. $F^{\mathrm{2m\ e}}_{n:r;i}$ comes from a divergent integral (see appendix) and $\epsilon = (4-D)/2$ is the dimensional regularization parameter. For $k=1$, the meaning of $(-t_i^{[k]})^{-\epsilon}$ needs to be clarified; one defines $(-t_i^{[1]})^{-\epsilon} = 0$.

[3] In our formulas, we will omit a numerical factor, usually called c_Γ, that depends only on the dimensional regularization parameter $\epsilon = (4-D)/2$ and is not relevant to our analysis.

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In (3.2) and (3.3), the amplitude is expressed as a sum of many dilogarithms — five for every box function. However, the first four dilogarithms in (3.3) can be eliminated, in favor of products of logarithms, when one performs the sum in (3.2). Thus, the total amplitude can be alternatively written as a sum involving only the fifth dilogarithm in (3.3) (plus products of logarithms). This simplified form for the amplitude was obtained in [1], and makes it possible to write the amplitude for $n \leq 5$ in terms of logarithms only. However, it turns out that to understand the twistor space structure of the amplitudes, it is better to work directly with the box functions.

From the definition of the box function (3.3), it follows that it is really just a function of three vectors. Two of them are p_{i-1} and p_{i+r}, the momenta of the two external gluons that are not combined. Of course, these vectors are lightlike. To simplify the following formulas, we call these vectors p and q. The third vector that enters the box function is the total momentum $P = p_i + p_{i+1} + \cdots + p_{i+r-1}$ of one set of combined gluons. We also set $Q = -p - q - P = p_{i+r+1} + p_{i+r+2} + \cdots + p_{i-2}$, the momentum of the other set of combined gluons, so momentum conservation is expressed as $p + q + P + Q = 0$.

Using the new notation, we define a generic scalar function as follows,

$$\begin{aligned} F(p,q,P) \, = \, & -\frac{1}{\epsilon^2}\left[(-(P+p)^2)^{-\epsilon} + (-(P+q)^2)^{-\epsilon} - (-P^2)^{-\epsilon} - (-Q^2)^{-\epsilon}\right] + \\ & + \mathrm{Li}_2\left(1 - \frac{P^2}{(P+p)^2}\right) + \mathrm{Li}_2\left(1 - \frac{P^2}{(P+q)^2}\right) + \\ & + \mathrm{Li}_2\left(1 - \frac{Q^2}{(Q+q)^2}\right) + \mathrm{Li}_2\left(1 - \frac{Q^2}{(Q+p)^2}\right) - \\ & - \mathrm{Li}_2\left(1 - \frac{P^2 Q^2}{(P+p)^2 (P+q)^2}\right) + \frac{1}{2}\log^2\left(\frac{(P+p)^2}{(P+q)^2}\right). \end{aligned} \tag{3.4}$$

The connection with $F^{\mathrm{2m\,e}}_{n:r;i}$ is achieved by simply taking $p = p_{i-1}$, $q = p_{i+r}$ and $P = p_i + \cdots + p_{i+r-1}$.

3.2 Decomposition of the amplitude

Our goal is to decompose $F(p,q,P)$ as a sum of functions whose twistor transforms are localized on simple algebraic sets. This will be analogous to the representation of tree amplitudes as a sum of MHV tree diagrams, obtained in [4]. The decomposition will not be manifestly Lorentz covariant. As in [4], to make the decomposition, we must define a positive chirality spinor λ_P for an arbitrary momentum vector P, not necessarily lightlike. We do this by introducing an arbitrary negative chirality spinor $\eta^{\dot{a}}$ (which we take to be the same for all P) and setting

$$\lambda_{P\,a} = P_{a\dot{a}}\eta^{\dot{a}}\,. \tag{3.5}$$

We also adopt this definition for a lightlike vector p, i.e., $\lambda_{p\,a} = p_{a\dot{a}}\eta^{\dot{a}}$. This is compatible with but more precise than the usual definition; usually, for lightlike p, λ_p is defined up to scaling by requiring that $p_{a\dot{a}} = \lambda_{p\,a}\widetilde{\lambda}_{p\,\dot{a}}$ for some $\widetilde{\lambda}_{p\,\dot{a}}$. The virtue of breaking this scaling symmetry and using (3.5) for all momenta, lightlike or not, is that it ensures identities like $\langle p, P+q\rangle = \langle p, P\rangle + \langle p, q\rangle$.

We define the inner product

$$\langle P, Q\rangle = \epsilon_{ab}\lambda_P^a\lambda_Q^b \tag{3.6}$$

for any momenta P and Q, lightlike or not.

With these definitions, we can factorize the argument of the fifth dilogarithm in (3.4), using the following rather unexpected identity:

$$1 - \frac{P^2Q^2}{(P+p)^2(P+q)^2} = \frac{(1-x)(1-y)}{xy} \tag{3.7}$$

where

$$x = \frac{\langle p, P\rangle}{\langle p, P+q\rangle}\frac{(P+q)^2}{P^2}\,, \qquad y = \frac{\langle q, P\rangle}{\langle q, P+p\rangle}\frac{(P+p)^2}{P^2}\,. \tag{3.8}$$

This identity holds for any choice of η. To prove the identity, note first of all that $\widetilde{\lambda}_p^{\dot{a}}$ and $\widetilde{\lambda}_q^{\dot{a}}$ give a basis for the space of negative chirality spinors, so we can assume that $\eta^{\dot{a}} = \alpha\widetilde{\lambda}_p^{\dot{a}} + \beta\widetilde{\lambda}_q^{\dot{a}}$ for some scalars α, β. Moreover, if $(\alpha,\beta) = (1,0)$, then (3.7) is straightforwardly verified using $\langle p, P\rangle = 2p\cdot P$, $\langle p, P+q\rangle = 2p\cdot(P+q)$, and $\langle q, P\rangle = \langle q, P+p\rangle$. So it suffices to show that the right hand side of (3.7) is independent of α and β. For any vector $P_{a\dot{a}}$ and spinors τ^a, $\widetilde{\nu}^{\dot{a}}$, we set $\langle\tau|P|\widetilde{\nu}] = \tau^a P_{a\dot{a}}\widetilde{\nu}^{\dot{a}}$. Then we compute

$$\frac{1-x}{x} = \frac{1}{(P+q)^2}\frac{\alpha(2p\cdot qP^2 - 4P\cdot pP\cdot q) + \beta(-2P\cdot q\,\langle p|P|q])}{2\alpha p\cdot P + \beta\langle p|P|q]}\,, \tag{3.9}$$

and likewise

$$\frac{1-y}{y} = \frac{1}{(P+p)^2}\frac{\beta(2p\cdot qP^2 - 4P\cdot pP\cdot q) + \alpha(-2P\cdot p\,\langle q|P|p])}{\alpha\langle q|P|p] + 2\beta P\cdot q}\,. \tag{3.10}$$

Let us write $(1-x)/x = (s\alpha+t\beta)/(u\alpha+v\beta)$ and $(1-y)/y = (s'\alpha+t'\beta)/(u'\alpha+v'\beta)$, with coefficients s, t, etc., that are independent of α and β. We claim that $(s\alpha+t\beta)/(u'\alpha+v'\beta)$ is independent of α and β, as is $(s'\alpha+t'\beta)/(u\alpha+v\beta)$. The claim clearly implies that $(1-x)(1-y)/xy$ is independent of α and β, as desired. For example, the condition for $(s\alpha+t\beta)/(u'\alpha+v'\beta)$ to be independent of α and β is $sv'-tu' = 0$. In the present context, this condition becomes

$$2p\cdot qP^2 - 4P\cdot pP\cdot q + \langle p|P|q]\langle q|P|p] = 0\,. \tag{3.11}$$

This identity — which if the spinors are written in conventional notation would be called a Fierz identity — holds for any lightlike vectors p, q and any vector P. The other condition we need, namely $s'v - t'u = 0$, follows from the same identity.

The factorization (3.7) is useful because it enables us to use Abel's dilogarithm identity:

$$\begin{aligned}\mathrm{Li}_2\left[\frac{(1-x)(1-y)}{xy}\right] &= \mathrm{Li}_2\left(\frac{1-x}{y}\right) + \mathrm{Li}_2\left(\frac{1-y}{x}\right) - \\ &\quad -\mathrm{Li}_2(1-x) - \mathrm{Li}_2(1-y) - \log x\log y\,.\end{aligned} \tag{3.12}$$

We can get more insight by means of further identities analogous to (3.7). By momentum conservation

$$1-\frac{P^2Q^2}{(P+p)^2(P+q)^2}=1-\frac{P^2Q^2}{(Q+q)^2(Q+p)^2}\,. \tag{3.13}$$

So upon introducing the variables related to x and y by exchange of P and Q

$$\widetilde{x}=\frac{\langle p,Q\rangle}{\langle p,Q+q\rangle}\frac{(Q+q)^2}{Q^2}\,,\qquad \widetilde{y}=\frac{\langle q,Q\rangle}{\langle q,Q+p\rangle}\frac{(Q+p)^2}{Q^2}\,, \tag{3.14}$$

we have

$$\frac{(1-x)(1-y)}{xy}=\frac{(1-\widetilde{x})(1-\widetilde{y})}{\widetilde{x}\widetilde{y}}\,. \tag{3.15}$$

In addition,

$$\frac{1-x}{y}=-\frac{1-\widetilde{x}}{\widetilde{x}}\,,\qquad \frac{1-y}{x}=-\frac{1-\widetilde{y}}{\widetilde{y}}\,. \tag{3.16}$$

Only one of these two identities requires a proof since the other one follows from the first and (3.15).

The proof of (3.16) goes along the same lines as the proof of (3.7). First, the expression

$$\left(\frac{\widetilde{x}}{1-\widetilde{x}}\right)\left(\frac{1-x}{y}\right) \tag{3.17}$$

is readily shown to equal -1 if $(\alpha,\beta)=(1,0)$. We show that it is independent of α and β as follows. We use the definitions (3.8) and (3.14), together with momentum conservation, to find

$$\left(\frac{\widetilde{x}}{1-\widetilde{x}}\right)\left(\frac{1-x}{y}\right)=\left(\frac{\langle p,P+q\rangle P^2-\langle p,P\rangle(P+q)^2}{\langle q,P\rangle}\right)\left(\frac{\langle q,Q\rangle}{\langle p,Q+q\rangle Q^2-\langle p,Q\rangle(Q+q)^2}\right). \tag{3.18}$$

The first factor on the right equals $(s\alpha+t\beta)/(u'\alpha+v'\beta)$ times a factor that is trivially independent of α and β (here s,t,u', and v' are as before), and so is independent of α and β. The second factor is also independent of α and β, since it can be obtained from the first by exchanging P and Q.

Having proven that

$$\frac{1-x}{y}=1-\frac{1}{\widetilde{x}}\,,\qquad \frac{1-y}{x}=1-\frac{1}{\widetilde{y}}\,, \tag{3.19}$$

we can use Landen's identity[4]

$$\mathrm{Li}_2(1-z)+\mathrm{Li}_2\left(1-\frac{1}{z}\right)=-\frac{1}{2}\log^2(z) \tag{3.20}$$

on $\mathrm{Li}_2\,(1-x)$ and $\mathrm{Li}_2\,(1-y)$ in (3.12) to get

$$\begin{aligned}\mathrm{Li}_2\left[\frac{(1-x)(1-y)}{xy}\right]&=\mathrm{Li}_2\left(1-\frac{1}{\widetilde{x}}\right)+\mathrm{Li}_2\left(1-\frac{1}{\widetilde{y}}\right)+\mathrm{Li}_2\left(1-\frac{1}{x}\right)+\\&\quad+\mathrm{Li}_2\left(1-\frac{1}{y}\right)+\frac{1}{2}\log^2\left(\frac{x}{y}\right).\end{aligned} \tag{3.21}$$

Note that by momentum conservation $x/y=\widetilde{y}/\widetilde{x}$ and therefore the right hand side of (3.21) is symmetric under exchanging P and Q or p and q.

[4]This form of Landen's identity is valid for $z>0$. For $z<0$ one has to add $-2\pi i\log(1-z)$.

If we use (3.21) to re-express the fifth dilogarithm in (3.4), each of the four dilogarithms on the right hand side of (3.21) can naturally be grouped with one of the first four dilogarithms already present in (3.4). We get an expression with eight dilogarithms, of which two are

$$-\mathrm{Li}_2\left(1-\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\frac{P^2}{(P+q)^2}\right)+\mathrm{Li}_2\left(1-\frac{P^2}{(P+q)^2}\right), \tag{3.22}$$

while the others are obtained from (3.22) by exchange of p and q and and/or exchange of P and Q.

It turns out that a particular combination of (3.22) with some terms involving products of logarithms gives a function whose twistor transform is supported on a simple algebraic set. Such a combination is

$$\begin{aligned} H_q(p,P) \;=\; & -\mathrm{Li}_2\left(1-\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\frac{P^2}{(P+q)^2}\right)+\mathrm{Li}_2\left(1-\frac{P^2}{(P+q)^2}\right)+ \\ & +\log\left(\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\right)\log\left(\frac{(P+q)^2}{\mu^2}\right)-\frac{1}{4}\log^2\left(\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\right). \end{aligned} \tag{3.23}$$

where μ is an arbitrary scale. Introducing such a scale is natural for the divergent terms in the box function but somewhat unnatural for a function that contributes only to the finite part. However, it is easy to check that in the combination $H_q(p,P)+H_p(q,P)+H_q(p,Q)+H_p(q,Q)$ the μ dependence cancels. Moreover, after some algebra, we find that the box function can be written

$$\begin{aligned} F(p,q,P) \;=\; & -\frac{1}{\epsilon^2}\left[(-(P+p)^2)^{-\epsilon}+(-(P+q)^2)^{-\epsilon}-(-P^2)^{-\epsilon}-(-Q^2)^{-\epsilon}\right]+ \\ & +H_q(p,P)+H_p(q,P)+H_q(p,Q)+H_p(q,Q)-\log\left(\frac{\langle p,Q\rangle}{\langle p,P\rangle}\right)\log\left(\frac{\langle q,P\rangle}{\langle q,Q\rangle}\right). \end{aligned} \tag{3.24}$$

As we will now explain, this formula gives a convenient way to understand the twistor space support of the scattering amplitude.

3.3 Twistor interpretation

There are three different building blocks in the decomposition of the box function (3.24). They are of the form

$$-\frac{1}{\epsilon^2}(-P^2)^{-\epsilon}, \qquad H_q(p,P), \qquad \text{and} \qquad \log\left(\frac{\langle p,Q\rangle}{\langle p,P\rangle}\right)\log\left(\frac{\langle q,P\rangle}{\langle q,Q\rangle}\right). \tag{3.25}$$

The twistor transform of $-\frac{1}{\epsilon^2}(-P^2)^{-\epsilon}$, where $P=p_i+p_{i+1}+\ldots+p_{j-1}+p_j$ for some i and j, is localized on two lines L and L' that generically are disjoint. That the gluons in the set $\{i,i+1\ldots,j\}$ are contained in a line L is clear from the criterion stated at the end of section 2.1: the amplitude only depends on the sum of their momenta. Likewise, the other gluons $\{j+1,j+2,\ldots,i-1\}$ are contained in a second line L'. Generically, L and L' do not intersect; they are contained in no common plane (see figure 5a). This can be checked by using the operator K_{klmn} defined in section 2, with k,l corresponding to two gluons in L and m,n two gluons in L'. It is not difficult to prove by hand that no power of this operator annihilates $-\frac{1}{\epsilon^2}(-P^2)^{-\epsilon}$.

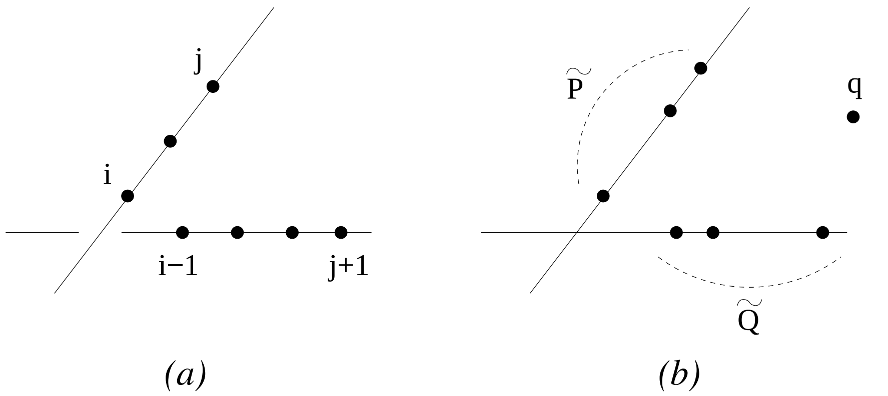

Figure 5: Twistor configurations contributing to the box function. (a) Two disjoint lines. (b) Two intersecting lines with q in the plane. Hypothetical twistor space propagators are not shown.

We now turn to $H_q(p,P)$. We write $\widetilde{P}$ for the set of gluons whose momenta adds to P, and $\widetilde{Q}$ for the set of gluons whose momenta adds to $Q+p$. The twistor transform of $H_q(p,P)$ is localized on a configuration in which gluons in $\widetilde{P}$ are contained in one line L, while gluons in $\widetilde{Q}$ are contained in another line L'. Moreover, the two lines L and L' intersect and so lie on a plane. The remaining gluon q is contained in this plane (with derivative of delta function support). See figure 5b.

The statement about the collinearity of gluons in $\widetilde{P}$, and likewise for collinearity of gluons in $\widetilde{Q}$, follows from the criterion stated at the end of section 2.1. Indeed, the dependence of the amplitude on gluons in $\widetilde{P}$ is only via the sum of the momenta; the dependence on gluons in $\widetilde{Q}$ is only via the holomorphic spinor λ_p and the sum of the momenta.

That the two lines L and L' intersect is a much less trivial fact. Indeed, appendix B is devoted to a proof of it. The proof is made by showing that any set of two points $P_1,P_2\in\widetilde{P}$ and two points $Q_1,Q_2\in\widetilde{Q}$ are coplanar. This is tested by using the coplanar operator and showing that

$$K_{P_1,P_2,Q_1,Q_2}H_q(p,P)=0\,. \tag{3.26}$$

It is also true that q is contained (with derivative of delta function support) in the plane containing L and L'. To show this, one has to check that $H_q(p,P)$ is annihilated by K^2, that is by any product $K_{ijkl}K_{i'j'k'l'}$. If any of the two K's does not contain q, then already a single K annihilates the amplitude. ($K_{ijkl}H_p(p,P)=0$ according to (3.26) if two of the points i,j,k,l are in $\widetilde{P}$ and two in $\widetilde{Q}$; if three or more are contained in $\widetilde{P}$ or $\widetilde{Q}$, it vanishes because three collinear points and any fourth point lie in a plane.) So the key case is that q is contained in $\{i,j,k,l\}$ and also in $\{i',j',k',l'\}$. We have not been able to find an analytic proof that $K_{ijkq}K_{i'j'k'q}H_q(p,P)=0$. However, we have verified this with computer assistance for up to seven gluons.

Finally, consider the remaining logarithmic term in (3.24):

$$\log\left(\frac{\langle p,Q\rangle}{\langle p,P\rangle}\right)\log\left(\frac{\langle q,P\rangle}{\langle q,Q\rangle}\right). \tag{3.27}$$

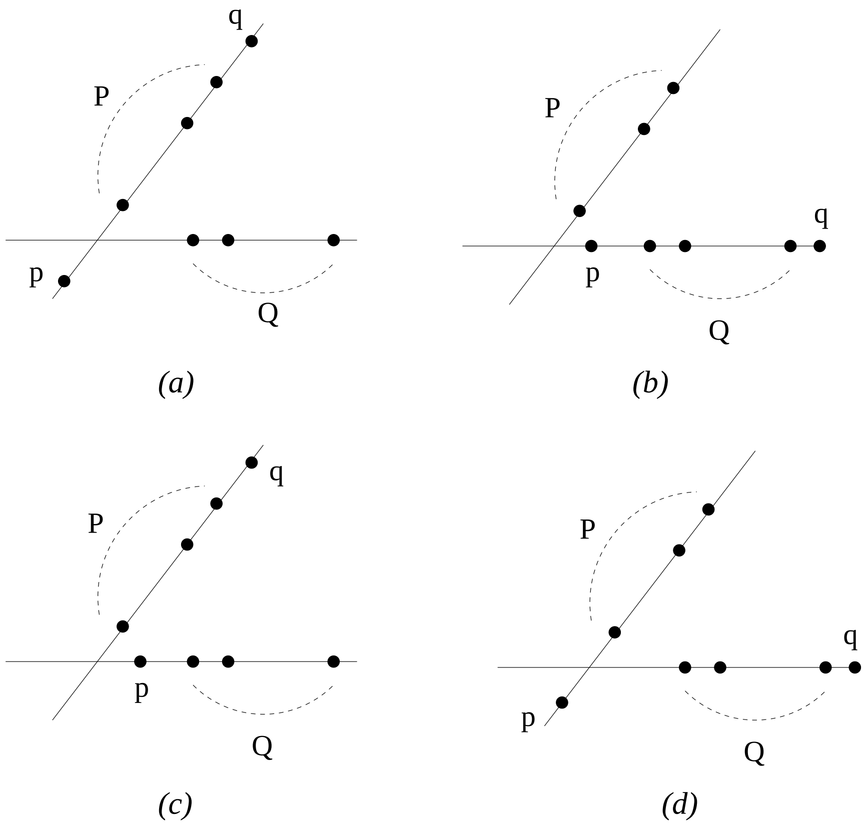

Figure 6: Twistor support of (a) $\log\langle p,Q\rangle\log\langle q,Q\rangle$, (b) $\log\langle p,P\rangle\log\langle q,P\rangle$, (c) $\log\langle p,Q\rangle\log\langle q,P\rangle$, and (d) $\log\langle p,P\rangle\log\langle q,Q\rangle$.

The twistor transform of (3.27) is localized on a plane. Indeed, any function of only the inner products $\langle\lambda,\lambda'\rangle$ of positive chirality spinors (of on-shell and off-shell momenta) is localized on a plane. One way to see this is again by showing that any operator K_{ijkl} annihilates it. From the definition of K in section 2, it is schematically given by $K = \lambda^1\lambda^2\mu^{\dot{1}}\mu^{\dot{2}}$. Therefore, under the twistor transform $\mu \to i\partial/\partial\widetilde{\lambda}$, K becomes a differential operator of degree two such that each term contains one derivative with respect to the $\widetilde{\lambda}^{\dot{1}}$ component of a gluon and one with respect to the $\widetilde{\lambda}^{\dot{2}}$ component of another gluon. Then for $\eta^{\dot{a}} = \delta^{\dot{a}\dot{2}}$, it is easy to see that any function of inner products $\langle\lambda,\lambda'\rangle$ is independent of the $\widetilde{\lambda}_{\dot{1}}$ component of all the gluons, and so is annihilated by K. Obviously, this conclusion does not depend on the choice of η.

But more is true. Expanding (3.27), we find four terms such as $\log\langle p,Q\rangle\log\langle q,Q\rangle$. Each of them is localized on two intersecting lines. The precise distribution of gluons can be deduced from the criterion at the end of section 2.1 and is shown in figure 6.

3.4 Concluding remarks

So we have determined the twistor space structure of the one-loop MHV amplitudes with $\mathcal{N}=4$ supersymmetry. But a few remarks are in order.

The decomposition that we have made is good enough to determine the twistor space structure of the amplitude, but a natural evaluation of these amplitudes in a suitable twistor-string theory might lead to a somewhat different decomposition. For example, the definition of $H_q(p, P)$ could be modified by adding some logarithmic terms.

The twistor transform of $H_q(p, P)$ describes two collinear sets of gluons, $\widetilde{P}$ and $\widetilde{Q}$, with an exceptional gluon (of momentum p) in the set $\widetilde{Q}$. Obviously, in another box function contributing to the scattering amplitude (3.2), decomposed in the same fashion, there is a contribution with the same collinear sets $\widetilde{P}$ and $\widetilde{Q}$, but the exceptional gluon contained in $\widetilde{P}$. In a suitable twistor-string computation, a single diagram might give the sum of these two contributions.

After removing the infrared-divergent part, which is supported on a pair of disjoint lines in twistor space, we have written the infrared-finite one-loop amplitude as a sum over subamplitudes (essentially the $H_q(p, P)$) associated with choices of how to combine a subset of adjacent gluons. Let $\mathcal{S}$ be the set of such choices. For each $s \in \mathcal{S}$, let $\mathcal{D}_s$ be one of the differential operators (a suitable F_{ijk} or a suitable power of some K_{ijkl}) that annihilates this subamplitude. Then our decomposition of the one-loop infrared-finite MHV amplitudes makes manifest that these amplitudes are annihilated by the differential operators $\prod_{s\in\mathcal{S}} \mathcal{D}_s$. A considerable amount of computer-based inquiry indicates that no other products of F's and K's annihilate these amplitudes. Thus, adding up the subamplitudes does not lead to any further simplification of the twistor space structure.

See appendix C for a discussion of a covariant decomposition of the box function.

4. $\mathcal{N} = 1$ one-loop MHV amplitude

The remainder of this paper is devoted to an analysis of (some) one-loop amplitudes with external gluons and internal massless particles in theories with reduced supersymmetry. The internal particles in the loop may have spin 0, 1/2, or 1. A convenient basis for these three one-loop amplitudes is to consider the $\mathcal{N} = 4$ amplitude $A^{\mathcal{N}=4}$, which was the subject of section 3, the amplitude $A_{ch}^{\mathcal{N}=1}$ due to an $\mathcal{N} = 1$ chiral multiplet, which will be the subject of this section, and the amplitude A_{sc} with a scalar in the loop, which we consider in section 5.

4.1 $\mathcal{N} = 1$ MHV amplitude

$\mathcal{N} = 1$ supersymmetry — like $\mathcal{N} = 4$ — leads to the vanishing of gluon scattering amplitudes in which all external gluons or all but one have the same helicity. The first and simplest amplitude is thus the MHV amplitude with precisely two gluons of negative (or positive) helicity. These amplitudes have been computed for any number n of external gluons. The computations are made [2] by expressing the gluon scattering amplitudes in terms of scalar one-loop integrals with two, three, or four external lines; these are sometimes called bubble, triangle, or box integrals. We review the formulas here and then describe the twistor space structure in section 4.2.

In contrast to the $\mathcal{N} = 4$ one-loop MHV amplitudes, the analogous $\mathcal{N} = 1$ amplitudes depend nontrivially on which gluons have negative helicity. For an n-gluon process in which

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gluons i and j have negative helicity, the amplitude is [2]

$$
\begin{aligned}
A_{ch}^{\mathcal{N}=1} = A^{\rm tree}\times\Bigg\{ & \sum_{m=i+1}^{j-1}\sum_{s=j+1}^{i-1} b_{m,s}^{i,j} B(t_{m+1}^{[s-m]}, t_m^{[s-m]}; t_{m+1}^{[s-m-1]}, t_{s+1}^{[m-s-1]}) + \\
& + \sum_{m=i+1}^{j-1}\sum_{a=j}^{i-1} c_{m,a}^{i,j} \frac{\ln(t_{m+1}^{[a-m]}/t_m^{[a-m+1]})}{t_{m+1}^{[a-m]} - t_m^{[a-m+1]}} + \sum_{m=j+1}^{i-1}\sum_{a=i}^{j-1} c_{m,a}^{i,j} \frac{\ln(t_{a+1}^{[m-a]}/t_{a+1}^{[m-a-1]})}{t_{a+1}^{[m-a]} - t_{a+1}^{[m-a-1]}} + \\
& + \frac{c_{i+1,i-1}^{i,j}}{t_i^{[2]}} K_0(t_i^{[2]}) + \frac{c_{i-1,i}^{i,j}}{t_{i-1}^{[2]}} K_0(t_{i-1}^{[2]}) + \frac{c_{j+1,j-1}^{i,j}}{t_j^{[2]}} K_0(t_j^{[2]}) + \frac{c_{j-1,j}^{i,j}}{t_{j-1}^{[2]}} K_0(t_{j-1}^{[2]}) \Bigg\}.
\end{aligned}
\tag{4.1}
$$

Here $t_i^{[k]} = (p_i + p_{i+1} + \cdots + p_{i+k-1})^2$ for $k \geq 0$, and $t_i^{[k]} = t_i^{[n-k]}$ for $k < 0$. Sums are taken in cyclic order around the circle, so a sum $\sum_{k=i}^{j}$ is evaluated by summing over all k in the clockwise direction from i to j, regardless of whether i is greater than or less than j. Though we have not indicated this in writing the formula, the sum over a and m is restricted to $|a-m| > 1$, $|a+1-m| > 1$, so that the logarithms have a finite, nonzero argument.

The coefficients in front of the integral functions are

$$
\begin{aligned}
b_{m,a}^{i,j} &= 2\frac{\langle i,m\rangle\langle m,j\rangle\langle i,a\rangle\langle a,j\rangle}{\langle i,j\rangle^2\langle m,a\rangle^2} \\
c_{m,a}^{i,j} &= \frac{(\mathrm{tr}_+[\not p_i \not p_j \not p_m \not q_{m,a}] - \mathrm{tr}_+[\not p_i \not p_j \not q_{m,a} \not p_m])}{(p_i+p_j)^2}\frac{\langle i,m\rangle\langle m,j\rangle}{\langle i,j\rangle}\frac{\langle a,a+1\rangle}{\langle a,m\rangle\langle m,a+1\rangle},
\end{aligned}
\tag{4.2}
$$

where we have set $q_{i,j} = \sum_{l=i}^{j} p_l$ and where

$$
\mathrm{tr}_+[\not p_{a_1}\not p_{a_2}\not p_{a_3}\not p_{a_4}] = \frac{1}{2}\,\mathrm{tr}[(1+\gamma_5)\not p_{a_1}\not p_{a_2}\not p_{a_3}\not p_{a_4}] = [a_1a_2]\langle a_2a_3\rangle[a_3a_4]\langle a_4a_1\rangle\,. \tag{4.3}
$$

The function B comes from the scalar one-loop integral with two masses, shown in figure 7 and also discussed in appendix A. Using the conventions shown in the figure, $p = p_m$ and $q = p_s$ while $P = p_{m+1} + p_{m+2} + \cdots + p_{s-1}$ and $Q = p_{s+1} + p_{s+2} + \cdots + p_{m-1}$. So p, q, P, Q are the four incoming momenta of the box diagram. The scalar function B is the finite part[5] of the $\mathcal{N} = 4$ scalar box function (3.4) studied in section 3:

$$
\begin{aligned}
B(p,q,P,Q) &= F^{\rm finite}(p,q,P,Q) \\
&= \mathrm{Li}_2\left(1 - \frac{P^2}{(P+p)^2}\right) + \mathrm{Li}_2\left(1 - \frac{P^2}{(P+q)^2}\right) + \\
&\quad + \mathrm{Li}_2\left(1 - \frac{Q^2}{(Q+q)^2}\right) + \mathrm{Li}_2\left(1 - \frac{Q^2}{(Q+p)^2}\right) - \\
&\quad - \mathrm{Li}_2\left(1 - \frac{P^2Q^2}{(P+p)^2(P+q)^2}\right) + \frac{1}{2}\log^2\left(\frac{(P+p)^2}{(P+q)^2}\right).
\end{aligned}
\tag{4.4}
$$

[5]This is a slight imprecision in our language here since the divergent terms, proportional to $1/\epsilon^2$, in (3.4) also contain finite pieces which we do not include in B.

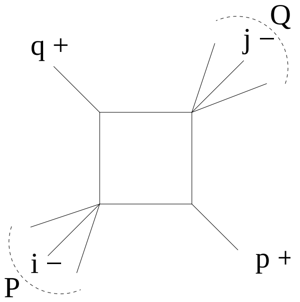

Figure 7: The scalar box integral contributing to the amplitude. Two of the vertices carry light-like momenta p and q. P and Q are sums of several light-like momenta. One negative-helicity gluon is in P and one is in Q; we label them as $i \in P$ and $j \in Q$.

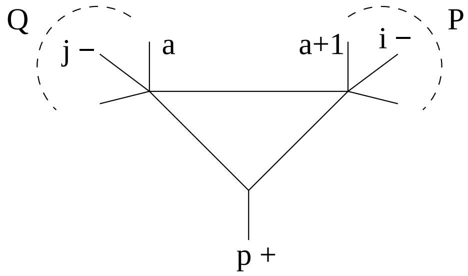

Figure 8: Triangle diagram contributing to the amplitude. p is a lightlike momentum, P is a sum of light-like momenta containing i and Q is a sum of momenta containing j.

The logarithmic terms in (4.1) arise from a scalar triangle diagram that is indicated in figure 8. It is convenient to write $p = p_m$ and Q and P for the sums $p_{m+1} + p_{m+2} + \cdots + p_a$ and $p_{a+1} + p_{a+2} + \cdots + p_{m-1}$. One of these sums contains p_i and one contains p_j; we write P for the sum that contains p_i and Q for the sum that contains p_j. Using the variables p, P, and Q, we can rewrite the complicated-looking expressions such as

$$\frac{\ln(t_{m+1}^{[a-m]}/t_m^{[a-m+1]})}{t_{m+1}^{[a-m]} - t_m^{[a-m+1]}} \tag{4.5}$$

in the convenient form

$$T(p, P, Q) = \frac{\ln(Q^2/P^2)}{Q^2 - P^2} \,. \tag{4.6}$$

The coefficient $c_{m,a}^{i,j}$ in (4.2) can be simplified using the definition (4.3):

$$\begin{aligned} c_{m,a}^{i,j} = {} & \frac{\langle i,m\rangle\langle m,j\rangle}{\langle i,j\rangle^2} \frac{\langle a,a+1\rangle}{\langle a,m\rangle\langle a,m+1\rangle} \times \\ & \times \begin{cases} (\langle j,m\rangle\langle i|P|m] + \langle i,m\rangle\langle j|P|m]) \,, & m = j+1, \ldots, i-1 \\ (\langle j,m\rangle\langle i|Q|m] + \langle i,m\rangle\langle j|Q|m]) \,, & m = i+1, \ldots, j-1. \end{cases} \end{aligned} \tag{4.7}$$

The main feature that we will use in remainder of the discussion is that the antiholomorphic dependence of the coefficients (4.7) (that is, their dependence on the negative chirality spinors $\widetilde{\lambda}$) is captured in p, P and Q. In particular, these coefficients are holomorphic in i, j, a.

The amplitude (4.1) diverges when gluon i or j becomes collinear with one of the adjacent positive helicity gluons, which we will label g. For $g = i-1$ or $g = i+1$, the piece that diverges when p_i and p_g become collinear can be evaluated in terms of a scalar bubble diagram that depends on $P = p_i + p_g$. It can be simplified to

$$\frac{c_{g,a}^{i,j}}{s_{i,g}} K_0(s_{i,g}) = -\frac{\langle i,g\rangle\langle g,j\rangle}{\langle i,j\rangle} \frac{\langle a,a+1\rangle}{\langle a,g\rangle\langle g,a+1\rangle} \frac{1}{\epsilon(1-2\epsilon)} (-P^2)^{-\epsilon} \,, \tag{4.8}$$

where $a = i, i-1$ for $g = i-1, i+1$ respectively. These terms are infrared-divergent, as is evident from the pole at $\epsilon = 0$; we write them as A_{IR}.

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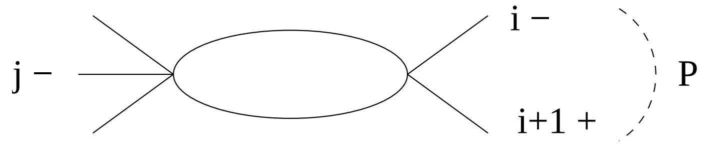


Figure 9: The scalar bubble diagram giving the divergent part $K_0(P^2)$ of the amplitude.

Collecting the different pieces, we can write the amplitude schematically as the sum of box, triangle and bubble contributions:

$$A_{ch}^{N=1} = A^{\rm tree} \times \left(\sum_{m,s;i\in P,j\in Q} b_{m,s}^{i,j} B(p_m, p_s, P, Q) + \sum_{m,a;i\in P,j\in Q} c_{m,a}^{i,j} T(p_m, P, Q) + A_{IR} \right). \tag{4.9}$$

4.2 Interpretation

Box diagrams. We first discuss the contribution to the amplitude from the scalar box functions,

$$b_{m,s}^{i,j} B(p, q, P, Q)\,. \tag{4.10}$$

Recall that we have introduced $p = p_m$ and $q = p_s$ in order to simplify the notation. The coefficient $b_{m,s}^{i,j}$ is a holomorphic function, that is a function only of the λ's, so it does not affect the twistor space structure of the amplitude. Hence, the localization of the box diagrams is determined by the the box function $B(p, q, P, Q)$. This is the finite part of the scalar box function (3.4), whose twistor-space structure was analyzed in section 3. It corresponds to twistor-space configurations in which the gluons whose momenta add to P are contained in one line L while the gluons whose momenta add to Q are contained in another line L' that is coplanar with L; moreover, of the remaining gluons p and q, one is contained in L or L' and one is contained in the plane containing L and L'.

There are some interesting differences between the two cases. For the $\mathcal{N} = 1$ chiral amplitude, of the two negative helicity gluons i and j, one is in L and one is in L'. (This follows from the details of the sum in (4.1).) For $\mathcal{N} = 4$, there is no such restriction: any two gluons may have negative helicity. Hence, for $\mathcal{N} = 1$, there is always precisely one negative helicity gluon on L and one on L', while for $\mathcal{N} = 4$, it is possible for both of these gluons to be on L or L', or for one to be on L or L' and the other to be in the bulk. We will give an intuitive explanation of this in section 4.3, after considering the other contributions to the chiral amplitude.

Triangle diagrams. Similarly, we can see part of the twistor space structure of the triangle amplitude

$$c_{m,a}^{i,j} T(p, P, Q) \tag{4.11}$$

without any additional work, using the criterion stated at the end of section 2.1. As the $\widetilde{\lambda}$'s only enter $c_{m,a}$ via p, P, and Q, the gluons whose momenta add to P are supported on a line L, and the gluons whose momenta add to Q are supported on another line L'.

Furthermore, all gluons are contained in a plane, that is a $\mathbb{CP}^2$, just as in the $\mathcal{N} = 4$ case. In fact, we found with some computer assistance that for up to seven gluons this amplitude is annihilated by K^2, that is by any product $K_{ijkl}K_{i'j'k'l'}$ of two collinear operators. The details of the resulting "derivative of a delta function support" on coplanar configurations are somewhat complicated, as one also has

$$K_{PPQQ}F_{pPP}F_{pQQ}(c^{i,j}_{m,a}T(p,P,Q)) = 0\,. \tag{4.12}$$

Here, K_{PPQQ} represents a coplanar operator K_{ijkl} with $i,j \in P$ and $k,l \in Q$. Similarly F_{pPP}, or F_{pQQ}, is a collinear operator F_{pij} with $i,j \in P$, or in Q, respectively. This means roughly that while it is possible to have a first order fluctuation away from coplanarity (since the triangle amplitude is annihilated by K^2 but not by K), either the two lines are strictly coplanar or one of them contains the point p.

Divergent part. Just as in the $\mathcal{N} = 4$ case, the infrared divergent part of the amplitude (4.1) is

$$\frac{1}{\epsilon(1-2\epsilon)}(-P^2)^{-\epsilon} \tag{4.13}$$

times a holomorphic function of spinors. As we discussed in section 3, it localizes on a disjoint union of lines. The gluons whose momenta add to P are on one line and the remaining gluons are on the second line.

4.3 Comparison of amplitudes

The surprising result of sections 3 and 4.2 is that at one-loop order, the $\mathcal{N} = 4$ MHV amplitude and the $\mathcal{N} = 1$ chiral MHV amplitude have very similar twistor space structure. (We will see in section 5.3 that the cut-constructible part of the nonsupersymmetric $--+++\ldots+$ amplitude also has a similar stucture.) In each case, apart from an elementary, infrared divergent contribution that localizes on two disjoint lines, we have found a novel twistor structure in which $n-1$ gluons are contained in a pair of intersecting lines, and the remaining gluon is in the plane that contains the lines.

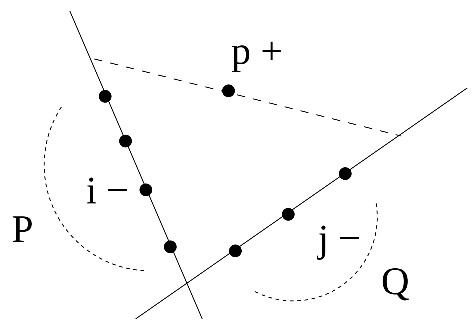


Figure 10: A twistor configuration contributing to the $N = 1$ chiral amplitude. All gluons except one are contained in a pair of intersecting lines; the last gluon is in the plane containing the lines. We suppose that this last gluon is connected to the lines by a twistor propagator, shown as a dashed line. The gluons i and j are localized on opposite lines, because only scalar and fermions can run around the loop.

Unfortunately, we do not know how a twistor-string theory would generate this structure. A guess is indicated in figure 10. (This discussion needs to be revisited in view of a holomorphic anomaly that will be discussed elsewhere.) Here we imagine two lines L and L' — understood as D-instantons — which intersection There is also a twistor space propagator connecting L and L'. One of the n gluons is attached to this propagator. We do not know why L and L' intersect or why the gluon attached to the propagator is contained in the same plane that contains L and L'. The lines L and L' together with the twistor propagator form a loop. Propagating around this loop is a particle of spin $1, 1/2$, or 0, in the case of the $\mathcal{N}=4$ amplitude, or $1/2$ or 0, in the case of the $\mathcal{N}=1$ chiral amplitude.

In this picture, the D-instantons are of degree one and represent MHV tree amplitudes, while the attachment of the n^{th} gluon to the twistor space propagator is presumably made using a local twistor space interaction (similar to the $\mathcal{A}^3$ twistor space interaction that comes from the Chern-Simons form [3]). From this picture, we can see why for $\mathcal{N}=4$, the negative helicity gluons can be inserted anywhere, while for $\mathcal{N}=1$ one of them is inserted on L and one on L'. In fact, for $\mathcal{N}=1$, the field propagating around the loop has helicity $\leq 1/2$. In a local, cubic interaction in twistor space, the three helicities add up to $+1$, so such an interaction does not couple a negative helicity gluon to fields of helicity $\leq 1/2$. This explains why for $\mathcal{N}=1$, a negative helicity gluon is not attached to the propagator in figure 10. The two negative helicity gluons cannot be attached to the same line L or L' for a similar reason: in a degree one tree level amplitude with k external fields of spin $1, 1/2$, or 0, the helicities must add up to $k-2$, which is not possible if two particles have helicity -1 and one has helicity $\leq 1/2$.

5. Nonsupersymmetric one-loop amplitudes

In this section, we explore the twistor structure of some nonsupersymmetric one-loop amplitudes. The amplitudes that we consider are scalar amplitudes, that is amplitudes with a massless scalar propagating in the loop. (As explained at the beginning of section 4, nonsupersymmetric amplitudes due to a massless field of spin $1/2$ or 1 are linear combinations of the scalar amplitudes with supersymmetric amplitudes that we have already studied.)

A conspicuous difference between supersymmetric amplitudes and nonsupersymmetric ones is that in the nonsupersymmetric case, there exist n-gluon amplitudes in which n or $n-1$ gluons have positive helicity. These amplitudes have been computed, and we begin with them. Then we consider the "MHV" amplitudes — a term which is not quite right in the nonsupersymmetric case – with $n-2$ gluons of positive helicity. These amplitudes are known less completely.

5.1 All plus helicity one-loop amplitudes

The one-loop amplitude for $n \geq 4$ gluons all of positive helicity is [10, 11]

$$A_n^{1-\mathrm{loop}}(+,\ldots,+) = -\frac{i}{48\pi^2} \sum_{1\leq i_1<i_2<i_3<i_4\leq n} \frac{\langle i_1,i_2\rangle[i_2,i_3]\langle i_3,i_4\rangle[i_4,i_1]}{\langle 1,2\rangle\langle 2,3\rangle\cdots\langle n,1\rangle}\,. \tag{5.1}$$

We can also write the amplitude in terms of the momenta and positive chirality spinors of external particles

$$A_n = -\frac{i}{96\pi^2} \sum_{1\leq i_1<i_2<i_3<i_4\leq n} \frac{s_{i_1i_2}s_{i_3i_4} - s_{i_1i_3}s_{i_2i_4} + s_{i_1i_4}s_{i_2i_4} - 4i\epsilon_{\mu\nu\lambda\rho}p_{i_1}^\mu p_{i_2}^\nu p_{i_3}^\rho p_{i_4}^\sigma}{\langle 1,2\rangle\langle 2,3\rangle\cdots\langle n,1\rangle}\,. \tag{5.2}$$

This amplitude is single-valued and free of cuts. Indeed, the discontinuities of the all-plus one-loop amplitude across a cut would be proportional to a product of tree amplitudes, at least one of which would have less than two gluons of negative helicity and so would vanish.

The twistor structure of this amplitude is clear. The amplitude is polynomial in the $\widetilde{\lambda}$'s, so it is supported (with derivative of a delta function support) on a line, that is, all gluons are contained in some $\mathbb{CP}^1$. A line in twistor space corresponds to a point in Minkowski spacetime. So, like the tree level MHV amplitude, the all-plus amplitude is a rough twistor space analog of a local interaction.

It is consequently tempting to guess that one should extend the concept of MHV tree diagrams [4] to add the one-loop all-plus amplitudes as new interaction vertices, in addition to the familiar tree level MHV vertices. Before testing this idea out in examples, let us make a few remarks about hypothetical Feynman diagrams with vertices of these two types. Consider first a diagram with d MHV tree vertices. Each vertex has two negative helicity gluons. To make a connected tree diagram, we must connect the vertices with $d-1$ propagators, absorbing $d-1$ negative helicity gluons (as each propagator absorbs a negative helicity gluon at one end). This leaves $d+1$ negative helicity external gluons. To make a diagram with l loops, we need l additional propagators, leaving

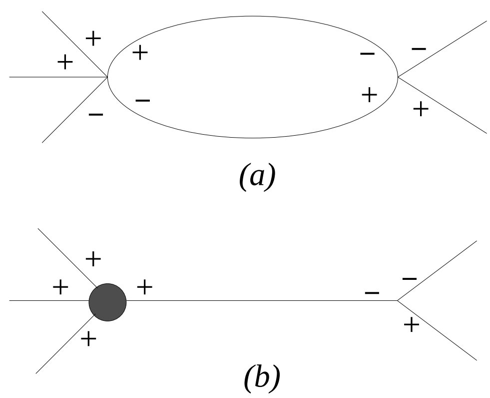


Figure 11: (a) A diagram with tree level MHV vertices. Each vertex couples to two negative helicity gluons; each propagator absorbs one such gluon. (b) A diagram that also contains a one-loop all-plus vertex, shown as a solid disc to symbolize the fact that a loop is hidden.

$$q = d + 1 - l \tag{5.3}$$

external gluons of negative helicity. This formula, which has already been explained in [4], is illustrated in figure 11a. While keeping the degree d fixed, we can replace an MHV tree vertex with a one-loop all-plus vertex; to also keep the number l of loops fixed, we should remove one propagator. See 11b for an example of such a diagram. In this process, the change in the vertex reduces q by 2, while removing a propagator increases it by 1. So overall, q is reduced by 1 and (5.3) becomes

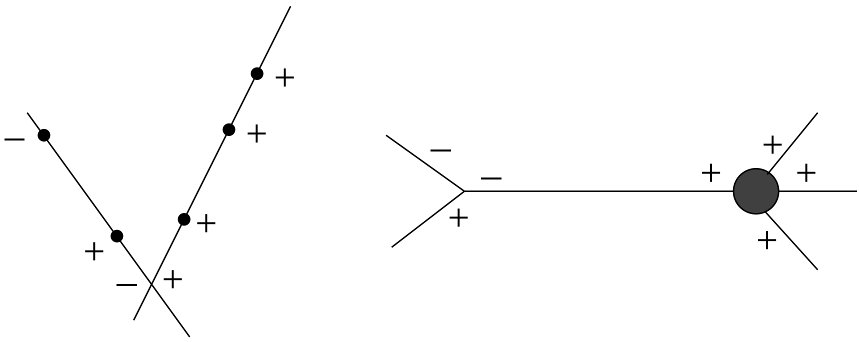
$$q = d + 1 - l - p\,, \tag{5.4}$$

where p is the number of all-plus vertices. The possible range of p is $0 \leq p \leq l$, so we can also write an inequality

$$q - 1 + l \leq d \leq q - 1 + 2l\,. \tag{5.5}$$

5.2 The $- + + \ldots + +$ one loop amplitude

Now we will compare this guess to the actual behavior of the nonsupersymmetric amplitudes with a single gluon of negative helicity. This comparison has revealed both good



(a) (b)

Figure 12: Two representations of a diagram contributing to a one-loop nonsupersymmetric amplitude with helicities $-++++$. (a) The twistor-space geometry as found from the differential equations. The two lines intersect as shown. (b) A hypothetical representation of the amplitude in terms of a tree diagram with a four-valent all-plus vertex and a three-valent MHV tree vertex.

news and bad news. The good news is that the twistor space structure agrees with what we would expect. The bad news is that we have not been able to find an off-shell continuation of the all-plus scattering amplitude to give the right amplitude.

The one-loop nonsupersymmetric scattering amplitudes in which all gluons but one have the same helicity have been derived by Mahlon [12, 13] using recursive techniques. For example, the five-gluon one-loop amplitude with helicities $-++++$ is [14, 15]

$$A = \frac{i}{48\pi^2}\frac{1}{\langle 34\rangle^2}\left[-\frac{[25]^3}{[12][51]} + \frac{\langle 14\rangle^3[45]\langle 35\rangle}{\langle 12\rangle\langle 23\rangle\langle 45\rangle^2} - \frac{\langle 13\rangle^3[32]\langle 42\rangle}{\langle 15\rangle\langle 54\rangle\langle 32\rangle^2}\right]. \tag{5.6}$$

The product of any three coplanar operators annihilates this amplitude

$$K^3 A = 0\,. \tag{5.7}$$

This can be proved as follows. Using the homogeneity of K in the $\partial/\partial\widetilde{\lambda}$'s, it follows that K^2A is homogeneous of degree -1 in inner products $[i,j]$ of negative helicity spinors. In section 3 of [3], it is shown that any momentum-conserving five-gluon amplitude, such as K^2A, that is homogeneous of degree -1 in the $[i,j]$, is annihilated by K. So $K^3A = 0$. We have also verified with computer assistance that A is annihilated by a certain product of collinear operators,

$$F^3_{234}F^3_{345}A = 0\,. \tag{5.8}$$

These statements correspond to the twistor space picture of figure 12a. (The picture must be infinitesimally thickened to allow for "derivative of a delta function support," because of the cubes in (5.7) and (5.8).) All five gluons are contained in a plane because of (5.7). In addition, (5.8) asserts that three of them (2, 3, and 4 or 3, 4, and 5 — in other words, three consecutive positive helicity gluons) are contained in a line. Drawing a straight line through the other two points, we find that the five gluons are on a union of two intersecting lines, as indicated in the figure. This figure is in agreement with what we would expect from the Feynman diagram of 12b, with an MHV vertex and an all-plus vertex (and a $1/P^2$ propagator, whose principal part describes an intersection of the two lines in twistor space, as we explained at the end of section 2). One subtlety is worth noting. In 12, all contributions have external helicities $+++$ on one line (or vertex) and $+-$ on the other; there is no contribution with $++$ on one side and $++-$ on the other side. This is what we would expect if the all-plus vertices are n-valent with $n \geq 4$, in other words if the all-plus vertex with $n = 3$ vanishes off-shell just as it does on-shell.

We have similarly studied the $-+++\ldots++$ amplitudes with up to eight gluons and found that they obey differential equations that are consistent with the twistor space picture of 12a (with the extra gluons added on one line or the other, preserving the cyclic order).

Off-shell continuation of the all-plus amplitude? On the other hand, we have not been successful in finding an off-shell continuation of the all-plus one-loop amplitudes to use in diagrams such as figure 12b. To find this off-shell continuation, one approach we considered was to take the second version (5.2) of the all-plus amplitude. This only depends on the momenta of external lines, which are defined off-shell, and on the positive chirality spinors λ, which we continued off-shell in [4]. So this gives a candidate off-shell continuation of the all-plus amplitude, but we have found that when inserted in Feynman diagrams such as that of 12b, it does not lead to the right scattering amplitudes. A similar problem in defining appropriate off-shell continuations for MHV gravity amplitudes was found in [16, 17].

An interesting point here is that given the absence of a one-loop $+++$ vertex, it is impossible to write a tree diagram with MHV and one-loop all-plus vertices that contribute to the one-loop $-+++$ amplitude. Yet the one-loop $-+++$ amplitude is nonzero. It therefore is necessary, from this point of view, to interpret the one-loop $-+++$ amplitude as a new local vertex. Indeed, it can be shown that this amplitude is annihilated by F^2 and so is supported on a line in twistor space; it is thus at least somewhat natural to interpret it as a new vertex.

5.3 Nonsupersymmetric "MHV" amplitudes

Here we consider the non-supersymmetric one-loop amplitudes with two gluons of negative helicity. We might call them "MHV" amplitudes, but in the nonsupersymmetric case this name does not fit well, since the amplitudes with less than two negative helicity gluons are also nonzero, as we have just reviewed.

These $--++\ldots+$ amplitudes contain cuts that can be determined from unitarity. According to [2], the cut-constructible part of the scalar one-loop amplitude with gluons 1 and 2 having negative helicity (and the others positive helicity) is

$$A_{sc,\,\mathrm{cut}} = \frac{1}{3}A^{\mathcal{N}=1} - \frac{c_\Gamma}{3}A^{\mathrm{tree}} \sum_{m=4}^{n-1} \frac{L_2\left(t_2^{[m-2]}/t_2^{[m-1]}\right)}{\left(t_1^{[2]}t_2^{[m-1]}\right)^3} \times \\ \times \mathrm{tr}_+[\not{k}_1\not{k}_2\not{k}_m\not{q}_{m,1}]\ \mathrm{tr}_+[\not{k}_1\not{k}_2\not{q}_{m,1}\not{k}_m]\left(\mathrm{tr}_+[\not{k}_1\not{k}_2\not{k}_m\not{q}_{m,1}] - \mathrm{tr}_+[\not{k}_1\not{k}_2\not{q}_{m,1}\not{k}_m]\right), \quad (5.9)$$

where

$$L_2(x) = \frac{\ln(x) - (x - 1/x)/2}{(1-x)^3}\,. \tag{5.10}$$

Setting $P = p_{m+1} + p_{m+2} + \cdots + p_1$ and $Q = p_2 + p_3 + \cdots + p_{m-1}$, and writing A^{tree} for the tree level MHV amplitude, the scalar one-loop amplitude becomes

$$\begin{aligned} A_{sc,\,\text{cut}} &= \frac{1}{3}A^{\mathcal{N}=1} - \frac{c_\Gamma}{3}\frac{A^{\text{tree}}}{\langle 1,2\rangle^3}\sum_{m=4}^{n-1}\frac{L_2(P^2/Q^2)}{(Q^2)^3}\times \\ &\quad\times\langle 1,m\rangle\langle 2,m\rangle\langle 1|P|m]\langle 2|P|m](\langle 1,m\rangle\langle 2|P|m] - \langle 2,m\rangle\langle 1|P|m])\,. \end{aligned} \tag{5.11}$$

(As in section 3, we write $\langle\lambda|P|\mu] = \lambda^a P_{a\dot a}\mu^{\dot a}$.) Now we have two different triangle functions,

$$T(p,P,Q) = \frac{\ln(Q^2/P^2)}{Q^2 - P^2}\,,\qquad \widetilde{T}(p,P,Q) = \frac{L_2(P^2/Q^2)}{(Q^2)^3}\,. \tag{5.12}$$

Schematically, the amplitude is a sum of triangle diagrams

$$A_{sc,\,\text{cut}} = \frac{1}{3}\sum_{m=4}^{n-1} c^{1,2}_{1,m}T(p_m,P,Q) + \sum_{m=4}^{n-1}\widetilde{c}^{\,1,2}_m\widetilde{T}(p_m,P,Q)\,, \tag{5.13}$$

where the first sum gives $\frac{1}{3}A^{\mathcal{N}=1}$ and $\widetilde{c}^{\,1,2}_m$ is the coefficient in front of $\widetilde{T}(p_m,P,Q)$ in (5.11).

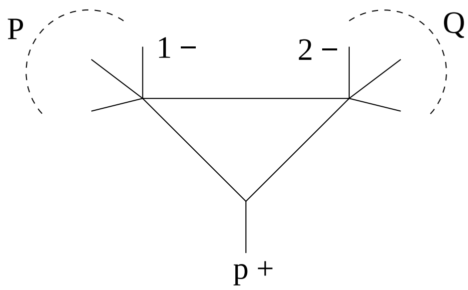


Figure 13: A triangle diagram contributing to the scalar loop amplitude with adjacent negative helicity gluons.

The first part of the amplitude, involving $T(p,P,Q)$, was studied in section 4. The part of the amplitude involving the nonsupersymmetric triangle function $\widetilde{T}(p,P,Q)$ has almost the same twistor space structure. The gluons whose momenta add to P are contained in a line L in view of the criterion of section 2.1, and the gluons whose momenta add to Q are likewise contained in a line L'. The $\widetilde{T}$ part of the amplitude is annihilated by K^2 (as we have found with some computer assistance), so all gluons are coplanar and in particular L and L' intersection Since the amplitude is annihilated by K^2 but not by K, it has "derivative of a delta function" support for such coplanar configurations; the nature of this "derivative of a delta function" support is further constrained by an additional equation that is analogous to (4.12):

$$K_{PPQQ}F^2_{pPP}F^2_{pQQ}\left(\widetilde{c}^{\,i,j}_{m,a}\widetilde{T}(p,P,Q)\right) = 0\,. \tag{5.14}$$

Cut-free terms. These cut-constructible terms do not give the full non-supersymmetric $--++\!+\ldots+$ amplitude. In particular, they lack poles in certain multiparticle channels. The missing parts of the amplitudes are cut-free rational functions. For five gluons with helicities $--+++$, the rational function has been computed via string-inspired methods [15].

With computer assistance, we have found that this rational function is annihilated by K^2, and so corresponds in twistor space to a planar configuration. Moreover, the rational

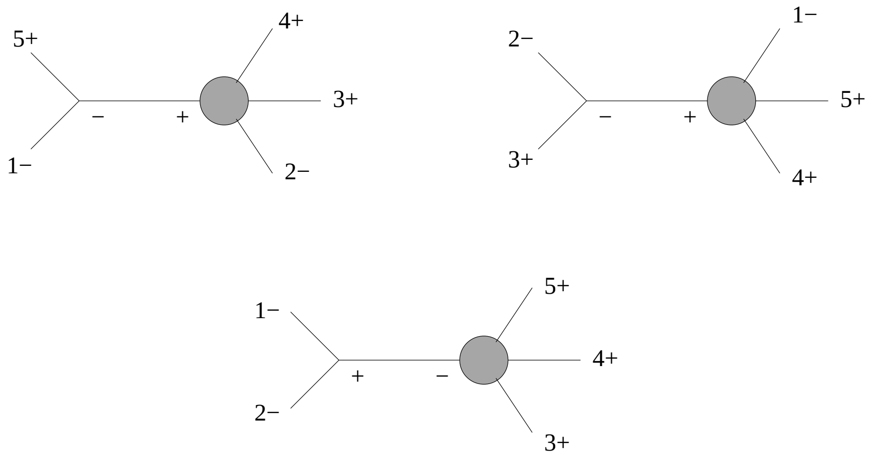


Figure 14: All possible diagrams contributing to the cut-free part of the full non-supersymmetric $--+++$ amplitude.

function is annihilated by $F^2_{234}F^2_{345}F^2_{451}$, and so is supported on configurations on which three gluons, including at most one of negative helicity, are collinear. It is not annihilated by $K_{1235}F^2_{234}F^2_{145}$, which (according to (5.14)) annihilates the cut-constructible part of the amplitude. These two facts can be understood if we assume again that there is no one-loop $+++$ vertex and that there is a one-loop $-+++$ vertex.

According to this, the cut-free part of the $--+++$ amplitude receives contributions from only three configurations, shown in figure 14. Each diagram has two vertices: one trivalent MHV vertex and one $-+++$ vertex. By definition, the trivalent MHV vertex must contain at least one external gluon of negative helicity and therefore the $-+++$ vertex has at most one external gluon of negative helicity. The collinear operators are squared because the $-+++$ vertex is localized on a line with a derivative of a delta function support. This also explains why $K_{1235}F^2_{234}F^2_{145}$ does not annihilate the amplitude. The collinear operators F^2_{234} and F^2_{145} annihilate the two configurations on the top in 14, leaving the one with gluons $3,4,5$ on the $-+++$ vertex. Due to the derivative of a delta function support, a single K_{1235} is not enough to annihilate the diagram and it should be supplemented by an extra collinear operator F_{345}.

The fact that the rational function part of the $--+++$ amplitude is annihilated by K^2 gives further evidence that there is no one-loop $+++$ vertex, even off-shell. If such a vertex exists, the one-loop nonsupersymmetric amplitude $--+++$ can receive a contribution from a tree diagram with two MHV vertices and one all-plus vertex (15). This configuration is not planar, so its contribution would not be annihilated by K^2.

Unfortunately, the cut-free parts of non-supersymmetric one-loop "MHV" amplitudes have not yet been computed for $n > 5$ gluons. These amplitudes should receive contributions from the quiver of figure 15 (with additional positive helicity gluons placed on the all-plus line). So their cut-free part should not be annihilated by K^2, but should obey differential equations reflecting the structure of this quiver.

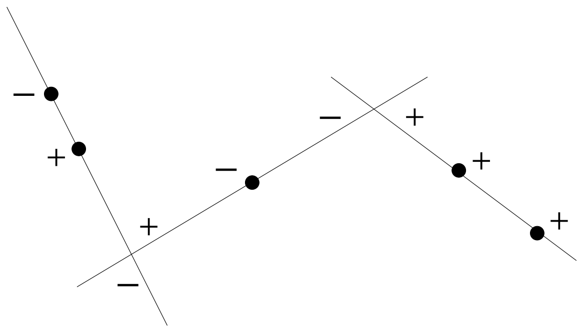


Figure 15: A diagram that could contribute to the rational function part of the $--+++$ loop amplitude if a $+++$ one-loop vertex is included.

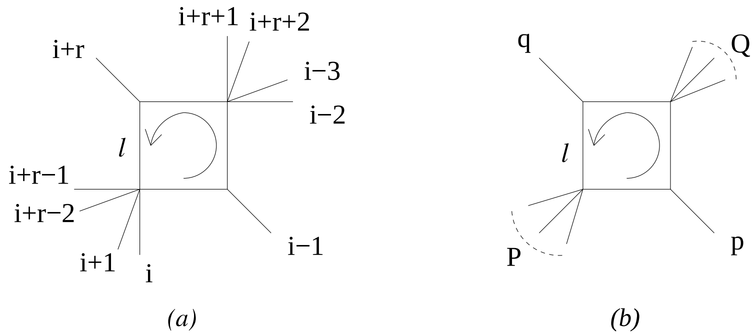


Figure 16: Scalar Box Integrals used in the definition of: (a) The box function $F^{\rm 2m\ e}_{n:r;i}$. (b) The generic box function $F(p,q,P)$.

Acknowledgments

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A. Box functions

The box function $F^{\rm 2m\ e}_{n:r;i}$ is one of a set of functions constructed from the scalar box integrals. The latter form a complete list of the possible integrals that can appear in a Feynman diagrammatic computation of one-loop amplitudes in $\mathcal{N}=4$ gauge theory.[6]

These integrals are known as the scalar box integrals because they would arise in a one-loop computation of a scalar field theory with four internal propagators.

The scalar box integral is defined as follows:

$$I_4 = -i(4\pi)^{2-\epsilon}\int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}}\frac{1}{\ell^2(\ell-K_1)^2(\ell-K_1-K_2)^2(\ell+K_4)^2}\,. \tag{A.1}$$

[6]After Passarino-Veltman reduction formulas are applied.

The incoming external momenta at each of the vertices are K_1, K_2, K_3, K_4. The labels are given in consecutive order following the loop. Momentum conservation implies that $K_1 + K_2 + K_3 + K_4 = 0$ and this is why (A.1) only depends on three momenta.

We are interested in the case when $K_1 = p_{i-1}$, $K_2 = p_i + \ldots + p_{i+r-1}$ and $K_3 = p_{i+r}$ (see figure 1a).

The box function (3.3) is then defined as follows,

$$F^{2me}_{n:r;i} = \left(t^{[r+1]}_{i-1} t^{[r+1]}_{i} - t^{[r]}_{i} t^{[n-r-2]}_{i+r+1}\right) I^{2me}_{4:r;i} \tag{A.2}$$

Here we follow the notation in [1], which is motivated by the fact that K_2 and K_4 are not lightlike and can be thought of as momenta of massive scalar particles. (2m e) stands for "two masses" and "easy." The "easy" case is when the two masses are at diagonally opposed corners. The "hard" case (2m h) is when the masses are adjacent. Fortunately, the latter does not enter in one-loop MHV amplitudes but it does for the six-gluon non-MHV one-loop amplitudes [2].

In section 3.1, we introduced the generic box function $F(p, q, P)$. This can be defined in a similar way by using the assignment of momenta shown in figure 1b.

B. Proof of coplanarity of lines in $H_q(p, P)$

The main result of section 3.2 was the decomposition of the box function (3.4),

$$\begin{aligned} F(p,q,P) \;=\; & -\frac{1}{\epsilon^2}\left[(-(P+p)^2)^{-\epsilon} + (-(P+q)^2)^{-\epsilon} - (-P^2)^{-\epsilon} - (-Q^2)^{-\epsilon}\right] + \\ & + H_q(p,P) + H_p(q,P) + H_q(p,Q) + H_p(q,Q) - \log\left(\frac{\langle p,Q\rangle}{\langle p,P\rangle}\right)\log\left(\frac{\langle q,P\rangle}{\langle q,Q\rangle}\right) \end{aligned} \tag{B.1}$$

where

$$\begin{aligned} H_q(p,P) \;=\; & -\mathrm{Li}_2\left(1 - \frac{\langle p, P+q\rangle}{\langle p,P\rangle}\frac{P^2}{(P+q)^2}\right) + \mathrm{Li}_2\left(1 - \frac{P^2}{(P+q)^2}\right) + \\ & + \log\left(\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\right)\log(P+q)^2 - \frac{1}{4}\log^2\left(\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\right). \end{aligned} \tag{B.2}$$

The twistor transform of the function $H_q(p, P)$ was shown to be localized on a configuration where gluons in $\widetilde{P}$ are on a line L, and gluons in $\widetilde{Q}$ are on another line L'. It was further claimed that the two lines intersect and moreover that the remaining gluon with momentum q is contained (with derivative of delta function support) in the plane defined by the two lines.

In this appendix, we provide a proof of the fact that the two lines intersection More precisely, we prove the equivalent statement that the points corresponding to two gluons in $\widetilde{P}$ and two gluons in $\widetilde{Q}$ are coplanar. That is, we prove that

$$K_{P_1,P_2,Q_1,Q_2} H_q(p, Q) = 0\,. \tag{B.3}$$

where K_{ijkl} is the differential operator of degree two obtained from the geometric condition of coplanarity (2.2). (P_1, P_2) are any to gluons in $\widetilde{P}$ and (Q_1, Q_2) are any two gluons in $\widetilde{Q}$.

Using conformal invariance, we can set the twistor space coordinates of Q_1 and Q_2 to be $Z_{Q_1} = (1, 0, 0, 0)$ and $Z_{Q_2} = (0, 1, 0, 0)$. This reduces K_{P_1,P_2,Q_1,Q_2} to

$$K = \epsilon^{\dot{a}\dot{b}} \frac{\partial^2}{\partial \widetilde{\lambda}^{\dot{a}}_{P_1} \partial \widetilde{\lambda}^{\dot{b}}_{P_2}} \,. \tag{B.4}$$

Before getting into the proof of (B.3), let us note some useful facts about the dilogarithm and its derivatives. Using the definition of the dilogarithm, $\mathrm{Li}_2(x) = -\int_0^x \log(1-z)dz/z$, it is easy to see that

$$\frac{d}{dx} \mathrm{Li}_2\,(1-x) = \frac{\log x}{1-x} \,. \tag{B.5}$$

Moreover, upon using the chain rule one can show that if $X = X(y, z)$, then any differential operator $\mathcal{O}$ that is homogeneous and degree two in $\partial/\partial y$ and $\partial/\partial z$ acts to produce,

$$\mathcal{O}(\mathrm{Li}_2\,(1-X)) = -\log X \ \mathcal{O}(\log(1-X)) + \cdots \tag{B.6}$$

where $\ldots$ denotes rational functions of X and its derivatives.

The proof of (B.3) will proceed in three steps. First, we prove that the terms in $KH_q(p, P)$ containing logarithms vanish. Second, we prove that the rational part of $K\mathrm{Li}_2\left(1 - \frac{P^2}{(P+q)^2}\right)$ is zero. And third, we prove that the rational functions from the remaining two terms in (B.2) cancel each other.

B.1 Logarithmic terms

Using (B.6) and (B.2), we find that

$$\begin{aligned} K\left[H_q(p,P)\right]_{\log} = {} & \log\left(\frac{\langle p, P+q\rangle}{\langle p, P\rangle}\frac{P^2}{(P+q)^2}\right) K\left[\log\left(1 - \frac{\langle p, P+q\rangle}{\langle p, P\rangle}\frac{P^2}{(P+q)^2}\right)\right] - \\ & -\log\left(\frac{P^2}{(P+q)^2}\right) K\left[\log\left(1 - \frac{P^2}{(P+q)^2}\right)\right] \\ & +\log\left(\frac{\langle p, P+q\rangle}{\langle p, P\rangle}\right) K\left[\log(P+q)^2\right] . \end{aligned} \tag{B.7}$$

We have not included the term where K acts on $\log\left(\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\right)$, since it vanishes. To see this note that η can be chosen to be $\eta^{\dot{a}} = \delta^{\dot{a}\dot{2}}$ or $\eta^{\dot{a}} = \delta^{\dot{a}\dot{1}}$ so that only one and the same component of $\widetilde{\lambda}_{P_1}$ and $\widetilde{\lambda}_{P_2}$ appears. But K, given in (B.4), only contains mixed terms.

The only way $K\left[H_q(p,P)\right]_{\log}$ can vanish is if the coefficient of each of the independent logarithms vanishes. From (B.7) we see that there are only two independent logarithms, namely, $\log\left(\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\right)$ and $\log\left(\frac{P^2}{(P+q)^2}\right)$.

The coefficient that multiplies the first is

$$K\left[\log\left(1 - \frac{\langle p, P+q\rangle}{\langle p, P\rangle}\frac{P^2}{(P+q)^2}\right)\right] + K\left[\log(P+q)^2\right] , \tag{B.8}$$

while the coefficient of the second is

$$K\left[\log\left(1 - \frac{\langle p, P+q\rangle}{\langle p, P\rangle}\frac{P^2}{(P+q)^2}\right)\right] - K\left[\log\left(1 - \frac{P^2}{(P+q)^2}\right)\right] . \tag{B.9}$$

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The task at hand is to prove that (B.8) and (B.9) are zero. However, note that (B.9) can be written as (B.8) minus

$$K\left[\log\left((P+q)^2 - P^2\right)\right]. \tag{B.10}$$

Therefore, after proving the vanishing of (B.8) we are left with proving that (B.10) is zero.

As a warm up, let us prove first that (B.10) is zero. The argument of the logarithm equals $2P\cdot q$. Let us write $P = P_1 + P_2 + \widehat{P}$. Note that $2P\cdot q = \langle P_1, q\rangle[P_1, q] + \langle P_2, q\rangle[P_2, q] + 2\widehat{P}\cdot q$ is linear in $\widetilde{\lambda}^{\dot a}_{P_1}$ and $\widetilde{\lambda}^{\dot a}_{P_2}$. This implies that $K(2P\cdot q) = 0$. Therefore,

$$K\left[\log\left(2P\cdot q\right)\right] = -\frac{1}{(2P\cdot q)^2}\epsilon^{\dot a\dot b}\frac{\partial}{\partial\widetilde{\lambda}^{\dot a}_{P_1}}(2P\cdot q)\frac{\partial}{\partial\widetilde{\lambda}^{\dot b}_{P_2}}(2P\cdot q) = -\frac{\langle P_1, q\rangle\langle P_2, q\rangle}{(2P\cdot q)^2}[q,q] = 0\,. \tag{B.11}$$

In order to prove that (B.8) vanishes, we first add to it zero in the form $K\left[\log\left(\frac{\langle q,P\rangle}{\langle q,P\rangle}\right)\right]$. Then, we combine the terms as follows

$$K\left[\log\left(\frac{\langle p,P\rangle(P+q)^2 - \langle p,P+q\rangle P^2}{\langle q,P\rangle}\right)\right] + K\left[\log\left(\frac{\langle q,P\rangle}{\langle p,P\rangle}\right)\right]. \tag{B.12}$$

We have encountered the argument of the first logarithms before; it is equal to $(s\alpha + t\beta)/(u'\alpha + v'\beta)$, which was shown to be independent of η in the proof of (3.7). Therefore we can choose $\eta = \widetilde{\lambda}_q$ to evaluate it. This gives $\langle p|P|q]$. On the other hand, the second term in (B.12) is trivially zero for the same reason that $K\left[\log\left(\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\right)\right]$ vanishes, as discussed above.

So we are left with proving that

$$K\left(\log\langle p|P|q]\right) = 0\,. \tag{B.13}$$

A computation similar to (B.11) reveals that this is also proportional to $[q,q] = 0$.

This concludes the proof of the vanishing of (B.7).

B.2 Rational terms

As discussed before, this part of the proof is divided into two computations. First, we prove that

$$K\left[\mathrm{Li}_2\left(1 - \frac{P^2}{(P+q)^2}\right)\right]_{\text{rational}} = 0\,, \tag{B.14}$$

where the subscript means the rational part. Sometimes we will simply write "r" instead of the whole "rational" subscript.

Clearly, the rational part is obtained when one derivative acts to produce a logarithm times a rational function and the second derivative acts on the logarithm to produce a rational function. More explicitly, we have

$$K\left[\mathrm{Li}_2\left(1 - \frac{P^2}{(P+q)^2}\right)\right]_{\text{rational}} \sim \epsilon^{\dot a\dot b}\frac{\partial}{\partial\widetilde{\lambda}^{\dot a}_{P_1}}\left(\frac{P^2}{(P+q)^2}\right)\frac{\partial}{\partial\widetilde{\lambda}^{\dot b}_{P_2}}\left(\frac{P^2}{(P+q)^2}\right) \tag{B.15}$$

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where $\sim$ means equal up to an irrelevant rational function. By writing $\frac{P^2}{(P+q)^2}$ as $1-\frac{2P\cdot q}{(P+q)^2}$ we find, after a straightforward computation similar to (B.11), that

$$\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\widetilde{\lambda}^{\dot{a}}_{P_1}}\left(\frac{2P\cdot q}{(P+q)^2}\right)\frac{\partial}{\partial\widetilde{\lambda}^{\dot{b}}_{P_2}}\left(\frac{2P\cdot q}{(P+q)^2}\right)=0\,. \tag{B.16}$$

This concludes the proof of (B.14).

Finally, we have to prove that

$$K\left[-\mathrm{Li}_2\left(1-\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\frac{P^2}{(P+q)^2}\right)+\log\left(\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\right)\log(P+q)^2\right]_{\mathrm{rational}}=0\,. \tag{B.17}$$

In principle, a direct computation of each term should provide a proof of (B.17). However, it turns out that the dilogarithm in (B.17) leads to a large proliferation of terms. Therefore, we seek an alternative way of computing it.

Let us start by using Landen's identity (3.20) on both dilogarithms of $H_q(p,P)$ to get

$$\begin{aligned}H_q(p,P)=\ &\mathrm{Li}_2\left(1-\frac{\langle p,P\rangle}{\langle p,P+q\rangle}\frac{(P+q)^2}{P^2}\right)-\mathrm{Li}_2\left(1-\frac{(P+q)^2}{P^2}\right)-\\ &-\log\left(\frac{\langle p,P\rangle}{\langle p,P+q\rangle}\right)\log P^2+\frac{1}{4}\log^2\left(\frac{\langle p,P\rangle}{\langle p,P+q\rangle}\right).\end{aligned} \tag{B.18}$$

For any linear and homogeneous second order differential operator $\mathcal{O}$ and any function $X=X(y,z)$, the following is true:

$$\mathcal{O}\left[\mathrm{Li}_2\left(1-\frac{1}{X}\right)\right]_{\mathrm{rational}}=-\frac{1}{X}\,\mathcal{O}\left[\mathrm{Li}_2\left(1-X\right)\right]_{\mathrm{rational}}\,. \tag{B.19}$$

This identity together with (B.14) implies that

$$K\left[\mathrm{Li}_2\left(1-\frac{(P+q)^2}{P^2}\right)\right]_{\mathrm{rational}}=0\,. \tag{B.20}$$

Therefore, applying K to both representations of $H_q(p,P)$ given by (B.2) and (B.18) and taking the rational part we conclude that

$$\begin{aligned}K&\left[\mathrm{Li}_2\left(1-\frac{\langle p\ P\rangle}{\langle p\ (P+q)\rangle}\frac{(P+q)^2}{P^2}\right)+\log\left(\frac{\langle p\ (P+q)\rangle}{\langle p\ P\rangle}\right)\log P^2\right]_{\mathrm{rational}}=\\ K&\left[-\mathrm{Li}_2\left(1-\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\frac{P^2}{(P+q)^2}\right)+\log\left(\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\right)\log(P+q)^2\right]_{\mathrm{rational}}=0\end{aligned} \tag{B.21}$$

where we have used that $K\left[\log^2\left(\frac{\langle p\ P\rangle}{\langle p\ (P+q)\rangle}\right)\right]=0$.

Using the identity (B.19) on the dilogarithm on the left we produce a dilogarithm with the same argument as the one on the right. Solving for it we find that

$$K\left[-\mathrm{Li}_2\left(1-\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\frac{P^2}{(P+q)^2}\right)\right]_{\mathrm{r}}=\frac{\alpha}{(\alpha-1)}K\left[\log\left(\frac{\langle p,P+q\rangle}{\langle p,P\rangle}\right)\log\left(\frac{(P+q)^2}{P^2}\right)\right]_{\mathrm{r}} \tag{B.22}$$

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with

$$\alpha = \frac{\langle p, P+q\rangle}{\langle p, P\rangle} \frac{P^2}{(P+q)^2} \,. \tag{B.23}$$

Using (B.22) to replace the complicated term with the dilogarithm in (B.17) by a product of logarithms, we find the equivalent but much simpler statement

$$\begin{aligned} \langle p, P+q\rangle P^2 K \left[\log\left(\frac{\langle p, P+q\rangle}{\langle p, P\rangle}\right) \log P^2\right]_{\mathrm{r}} &= \\ = \langle p, P\rangle (P+q)^2 K \left[\log\left(\frac{\langle p, P+q\rangle}{\langle p, P\rangle}\right) \log (P+q)^2\right]_{\mathrm{r}} &. \end{aligned} \tag{B.24}$$

This new identity can be checked straightforwardly by explicit computation.

C. Covariant decomposition of the box function

Our decomposition of the box function can be made manifestly covariant by choosing $\eta^{\dot a}$ to be the $\widetilde{\lambda}^{\dot a}$ of one of the external gluons. Clearly, different choices of $\eta^{\dot a}$ lead to very different looking formulas. There are two particularly interesting choices that reduce the total number of dilogarithms from eight to only four. In general there are four H-functions and each contains two dilogarithms. Choosing $\eta^{\dot a} = \widetilde{\lambda}^{\dot a}_p$ or $\eta^{\dot a} = \widetilde{\lambda}^{\dot a}_q$ sets to zero two of the H-functions. More explicitly we have:

$$\begin{aligned} \eta^{\dot a} = \widetilde{\lambda}^{\dot a}_p \;\; &\Rightarrow \;\; \frac{\langle q\ (P+p)\rangle}{\langle q\ P\rangle} = \frac{\langle q\ (Q+p)\rangle}{\langle q\ Q\rangle} = 1 \\ &\Rightarrow H_p(q,P) = H_p(q,Q) = 0, \eta^{\dot a} = \widetilde{\lambda}^{\dot a}_q \\ &\Rightarrow \frac{\langle p\ (P+q)\rangle}{\langle p\ P\rangle} = \frac{\langle p\ (Q+q)\rangle}{\langle p\ Q\rangle} = 1 \\ &\Rightarrow H_q(p,P) = H_q(p,Q) = 0\,. \end{aligned} \tag{C.1}$$

Consider the first choice, i.e., $\eta^{\dot a} = \widetilde{\lambda}^{\dot a}_p$. Then the box function (3.4) is given by

$$\begin{aligned} F(p,q,P) \;=\; & -\frac{1}{\epsilon^2}\left[(-(P+p)^2)^{-\epsilon} + (-(P+q)^2)^{-\epsilon} - (-P^2)^{-\epsilon} - (-Q^2)^{-\epsilon}\right] \\ & -\mathrm{Li}_2\left(1 + A\frac{(1-B)}{(1-A)}\right) - \mathrm{Li}_2\left(1 + B\frac{(1-A)}{(1-B)}\right) - \frac{1}{2}\log^2\left(-\frac{(1-A)}{(1-B)}\right) + \\ & +\mathrm{Li}_2\left(1-C\right) + \mathrm{Li}_2\left(1-D\right) + \frac{1}{2}\log^2\left(\frac{C}{D}\right) \end{aligned} \tag{C.2}$$

where

$$A = \frac{P^2}{(P+p)^2}\,, \qquad B = \frac{Q^2}{(Q+p)^2}\,, \qquad C = \frac{P^2}{(P+q)^2}\,, \qquad D = \frac{Q^2}{(Q+q)^2}\,. \tag{C.3}$$

Note that we have decreased the number of dilogarithm in the original form of the box function (3.4) by one.

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